

Cosmological Baryon Sound Waves Coupled with the Primeval Radiation

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The fluid equations for the baryon-electron system in an expanding universe are derived from the Boltzmann equation. The effect of the Compton interaction is taken into account properly in order to evaluate the photon-electron collisional term. As an application, the acoustic motions of the baryon-electron system after recombination are investigated. The effective adiabatic index γ is computed for sound waves of various wavelengths, assuming the perturbation amplitude is small. The oscillations are found to be damped when γ changes from between 1 (for an isothermal process) to $5/3$ (for an adiabatic process).

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I. INTRODUCTION

One of the biggest interest in astrophysics is to understand how structures in the universe are formed. Observations reveal that the universe is not simply homogeneous and isotropic but contains rich structures from stars to the large scale structure such as clusters of galaxies, super-clusters, the great wall and voids in it. Although the gravity itself does not have any particular scale, several physical scales associated with the structure formation, such as the Jeans scale, the Silk damping scale and the horizon scale at the matter-radiation equality epoch, are derived by the inclusion of an effect of primeval radiation [1–3]. Among them, the Jeans scale determines whether small density fluctuations can grow against pressure or not.

The linear evolution of small density perturbations has been well understood. Outside the Jeans scale, their evolution is described by the growing mode solution [4,5]. Assuming the spherical symmetry, we can follow the evolution of over-dense regions. The critical threshold of a density contrast to collapse is 1.69 in the Einstein-de Sitter universe and we can estimate the fraction of the collapsed mass by employing the extrapolation of the linear perturbation theory [6]. If the wave length of fluctuations is smaller than the Jeans scale, however, fluctuations cannot grow up to the critical threshold but begin to oscillate as an acoustic wave. Therefore an investigation of the time evolution of the Jeans scale is crucial in particular for the study what is the first collapsed object in the universe.

The Jeans scale is ruled by the sound speed c_s and the free fall time of the object or the expansion time of the universe. After the recombination of hydrogen atoms, the radiation pressure becomes ineffective and equations of state are described by the adiabatic index γ of baryonic matter. If the energy transfer between the baryon and the photon is efficient, we expect isothermal process, i.e., $\gamma = 1$. However, if the time scale of the energy transfer becomes longer than the oscillation time scale of the sound wave, the adiabatic $\gamma = 5/3$ must be achieved. These two γ 's give about factor two difference for the Jeans scales in mass. This point is crucial when we investigate the evolution of the baryon density perturbations smaller than the Jeans scale after recombination. Moreover, the temperature of gas is very close to the radiation temperature at the recombination epoch but gradually separates from it. Eventually, it does change the dependence on the red-shift z from $1+z$ to $(1+z)^2$. In order to obtain an accurate Jeans scale, we need to take into account all these details. In this paper, we formulate the energy transfer

between baryons, electrons and photons from the first principle, i.e., the Boltzmann equation. Similar treatment on photons has been done by Hu, Scott & Silk [7] and Dodelson & Jubas [8]. These works focused on the perturbations of photons which are massless particles in order to get anisotropies of the cosmic microwave background radiation. Here we formulate perturbation equations of baryon-electron fluid which is the massive particle system. The energy transfer between this fluid system and photons is coming through the Compton scattering. We study in detail the sound speed and the Jeans scale after the recombination. It is found that damping of the sound wave is efficient during the change of γ .

In §II, we describe the Boltzmann equation in the perturbed expanding universe. In §III, the equations of the baryon-electron system are derived based on the fluid approximation, by integrating the Boltzmann equation for the momentum space. The integrations of the collisional term which describes the Compton interaction between electrons and photons is summarized in §IV. The perturbation equations of the baryon-electron fluid are obtained in §V. In §VI, we investigate the acoustic motion of the baryon-electron fluid after the recombination in the expanding universe as an application of our formalism. §VII is devoted to summary and discussions. In appendix A, we consider the perturbation equation for the rate equation in order to complete the perturbed equations obtained in §V. In appendix B, we put summary of the physical scales for the cosmological baryon perturbations in the universe. The evolution of the matter temperature is summarized in appendix C.

We will work in units where $c = \hbar = k_B = 1$.

II. BOLTZMANN EQUATION

We write the perturbed space-time to the Friedmann-Robertson-Walker space-time in the Newtonian gauge as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(1 + 2\Psi) dt^2 + \left(\frac{a}{a_0}\right)^2 (1 + 2\Phi) \delta_{ij} dx^i dx^j, \quad (2.1)$$

with introducing the perturbed gravitational potential Ψ and the curvature perturbation Φ . a is the scale factor and suffix 0 indicates the present value. Note that we employ a flat orthogonal coordinate system besides the scale factor as indicated by the Kronecker's delta δ_{ij} for a background space. Ψ satisfies the Poisson equation

$$\nabla^2 \Psi = 4\pi G \rho \left(\frac{a}{a_0}\right)^2 \delta, \quad (2.2)$$

where ρ and δ are total background density and density fluctuation, respectively, and $\Phi = -\Psi$ when the anisotropic stress is negligible, e.g., in the matter-dominated era.

We write next the Boltzmann equation for the distribution function $f_{(\alpha)}(t, x^i, q^i)$ of the (α) -particle as

$$\frac{\partial f_{(\alpha)}}{\partial t} + \frac{\partial f_{(\alpha)}}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f_{(\alpha)}}{\partial q^i} \frac{dq^i}{dt} = C[f_{(\alpha)}], \quad (2.3)$$

with the collision term in the right hand side. This Boltzmann equation is written in terms of the momentum q^i measured by an observer in the cosmological rest frame. In order to solve this equation, we must rewrite the terms dx^i/dt and dq^i/dt in terms of x^i and q^i , which is obtained from the equation of motion of the (α) -particle. To obtain this equation, we write the 4-momentum in the locally orthonormal frame as (q^0, q^i) . Here the energy in this frame q^0 is defined as

$$q^0 := \sqrt{\mathbf{q}^2 + m_{(\alpha)}^2}, \quad (2.4)$$

where $\mathbf{q}^2 = \sum_i (q^i)^2$.

The equation of motion is given from the geodesic equation. However, the geodesic equation of the (α) -particle is commonly written in terms of the 4-momentum p^μ in the frame (2.1), which is defined by $p^\mu = dx^\mu/d\lambda$. Here $\lambda(d\lambda = ds/m_{(\alpha)})$ is the affine parameter. We use the fact that the 4-momentum (q^0, q^i) is related to p^μ , as follows,

$$q^i = \sqrt{g_{ii}} \frac{dx^i}{d\lambda} = \frac{a}{a_0} (1 + \Phi) p^i, \quad (2.5)$$

$$q^0 = \sqrt{-g_{00}} \frac{dx^0}{d\lambda} = (1 + \Psi) p^0. \quad (2.6)$$

Equations (2.5) and (2.6) give the following relations, up to the first order of Ψ and Φ ,

$$\frac{dx^i}{dt} = \frac{p^i}{p^0} = \frac{a_0}{a} (1 + \Psi - \Phi) \frac{q^i}{q^0}, \quad (2.7)$$

and

$$\frac{dq^i}{dt} = \frac{\dot{a}}{a} q^i + \frac{a}{a_0} \left(\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x^j} \frac{dx^j}{dt} \right) p^i + \frac{a}{a_0} (1 + \Phi) \frac{dp^i}{dt}, \quad (2.8)$$

where the over-dot denotes t -differentiation.

On the other hand, the geodesic equation in the leading order of the perturbation derives

$$\frac{dp^i}{dt} = - \left(\frac{a_0}{a} \right)^2 \left[2 \frac{a}{a_0} \frac{\dot{a}}{a} (1 - \Phi) q^i + 2 \frac{a}{a_0} \dot{\Phi} q^i + 2 \Phi_{,j} \frac{q^j q^i}{q^0} + \delta^{ij} \left(\Psi_{,j} q^0 - \Phi_{,j} \frac{\mathbf{q}^2}{q^0} \right) \right], \quad (2.9)$$

where $_{,i} = \partial/\partial x^i$. Hereafter we will omit δ^{ij} and will write as e.g., $\Psi_{,i} = \delta^{ij} \Psi_{,j}$. Inserting this to Eq.(2.8),

$$\frac{dq^i}{dt} = - \frac{a_0}{a} \left[\frac{a}{a_0} \frac{\dot{a}}{a} q^i + \frac{a}{a_0} \dot{\Phi} q^i + \Phi_{,j} \frac{q^j q^i}{q^0} + \Psi_{,i} q^0 - \Phi_{,i} \frac{\mathbf{q}^2}{q^0} \right]. \quad (2.10)$$

Now we can write down the left hand side of the Boltzmann equation (2.3) using the equations (2.7) and (2.10). If we employ the conformal time defined by $(a/a_0)d\eta \equiv dt$ instead of the proper time, it becomes as

$$\begin{aligned} \frac{\partial f_{(\alpha)}}{\partial \eta} + \frac{\partial f_{(\alpha)}}{\partial x^i} (1 - \Phi + \Psi) \frac{q^i}{q^0} \\ + \frac{\partial f_{(\alpha)}}{\partial q^i} \left[- \frac{a'}{a} q^i - \Phi' q^i - \Psi_{,i} q^0 - \Phi_{,j} \frac{q^j q^i}{q^0} + \Phi_{,i} \frac{\mathbf{q}^2}{q^0} \right] = \frac{a}{a_0} C[f_{(\alpha)}], \end{aligned} \quad (2.11)$$

where the prime denotes η -differentiation. Until now, we have assumed only the smallness of perturbations to the homogeneous background.

If we introduce a further assumption that the motion of the (α) -particle is non-relativistic, the Boltzmann equation (2.11) reduces to

$$\frac{\partial f_{(\alpha)}}{\partial \eta} + \frac{\partial f_{(\alpha)}}{\partial x^i} \frac{q^i}{m_{(\alpha)}} + \frac{\partial f_{(\alpha)}}{\partial q^i} \left[- \left(\frac{a'}{a} + \Phi' \right) q^i - \Psi_{,i} m_{(\alpha)} \right] = \frac{a}{a_0} C[f_{(\alpha)}], \quad (2.12)$$

where $v^i = q^i/m_{(\alpha)}$ and terms of $O(\mathbf{v}^2 \times \Psi \text{ or } \Phi)$ and $O(\mathbf{v}^3)$ are omitted. This is a familiar non-relativistic Boltzmann equation in the expanding universe, when the term proportional to Φ' is neglected.

III. FLUID APPROXIMATION

We consider a system of non-relativistic particles (baryons and electrons) and photons which interact only with the electrons. Neutral and ionized hydrogen atoms and neutral helium atoms are taken into account as baryonic components. We further take a single fluid approximation for this baryon and electron system since the time scale of the interaction between them is short enough. Under these assumptions, we here derive the fluid equation from the Boltzmann equation (2.12).

Since the particles are non-relativistic, we take the distribution function of the (α) -particle, where $\alpha = e, H$ and He for the electron, the hydrogen and the helium, as

$$f_{(\alpha)} = S n_{(\alpha)}(\mathbf{x}) \left(\frac{2\pi}{m_{(\alpha)} T_b(\mathbf{x})} \right)^{3/2} \exp \left[\frac{-(\mathbf{q}_{(\alpha)} - m_{(\alpha)} \mathbf{v}_b(\mathbf{x}))^2}{2m_{(\alpha)} T_b(\mathbf{x})} \right], \quad (3.1)$$

with a normalization of

$$\int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3} f_{(\alpha)} = S n_{(\alpha)}(\mathbf{x}). \quad (3.2)$$

Here $T_b(\mathbf{x})$ and $\mathbf{v}_b(\mathbf{x})$ are the temperature and the peculiar velocity of this fluid system which the suffix b denotes. Then it follows that

$$\int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3} q_{(\alpha)}^i f_{(\alpha)} = m_{(\alpha)} S n_{(\alpha)} v_b^i, \quad (3.3)$$

$$\int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3} q_{(\alpha)}^i q_{(\alpha)}^j f_{(\alpha)} = m_{(\alpha)}^2 S n_{(\alpha)} v_b^i v_b^j + m_{(\alpha)} S n_{(\alpha)} T_b \delta^{ij}, \quad (3.4)$$

$$\int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3} \mathbf{q}_{(\alpha)}^2 q_{(\alpha)}^i f_{(\alpha)} = m_{(\alpha)}^3 S n_{(\alpha)} v_b^2 v_b^i + 5m_{(\alpha)}^2 S n_{(\alpha)} T_b v_b^i. \quad (3.5)$$

Operating the following integration and the summation for the Boltzmann equation (2.12) with the distribution function (3.1),

$$\sum_{(\alpha)} m_{(\alpha)} \int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3}, \quad (3.6)$$

we obtain the continuity equation

$$\frac{\partial \rho_b}{\partial \eta} + 3 \left(\frac{a'}{a} + \Phi' \right) \rho_b + \frac{\partial}{\partial x^i} (\rho_b v_b^i) = 0 \quad (3.7)$$

for

$$\rho_b(\mathbf{x}) \equiv \sum_{(\alpha)} m_{(\alpha)} S n_{(\alpha)}(\mathbf{x}). \quad (3.8)$$

The collision terms should cancel out by the summation.

Operating the following integration and the summation to Eq.(2.3),

$$\sum_{(\alpha)} \int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3} q_{(\alpha)}^i, \quad (3.9)$$

we get the Euler equation

$$\frac{\partial(\rho_b v_b^i)}{\partial \eta} + 4 \left(\frac{a'}{a} + \Phi' \right) \rho_b v_b^i + \frac{\partial(\rho_b v_b^i v_b^j)}{\partial x^j} + \frac{\partial P}{\partial x^i} + \rho_b \frac{\partial \Psi}{\partial x^i} = \Delta V_{\text{Compton}}^i, \quad (3.10)$$

where

$$P \equiv \sum_{(\alpha)} S n_{(\alpha)} T_b, \quad (3.11)$$

and

$$\Delta V_{\text{Compton}}^i \equiv \frac{a}{a_0} \int \frac{d^3 \mathbf{q}_{(e)}}{(2\pi)^3} q_{(e)}^i C[f_{(e)}]_{\text{Compton}}. \quad (3.12)$$

The contributions from the collision term between the baryon and the electron should cancel out while the contribution from the Compton interaction between the electron and the photon, $\Delta V_{\text{Compton}}$, remains. The explicit form of $\Delta V_{\text{Compton}}^i$ is given in the next section.

Operating the following integration and the summation to Eq.(2.3),

$$\sum_{(\alpha)} \frac{1}{2m_{(\alpha)}} \int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3} \mathbf{q}_{(\alpha)}^2, \quad (3.13)$$

we get the energy equation

$$\begin{aligned} & \frac{\partial}{\partial \eta} \left(\rho_b \left(\frac{\mathbf{v}_b^2}{2} + h \right) \right) + 5 \left(\frac{a'}{a} + \Phi' \right) \rho_b \left(\frac{\mathbf{v}_b^2}{2} + h \right) + \frac{\partial}{\partial x^i} \left(v_b^i \rho_b \left(\frac{\mathbf{v}_b^2}{2} + h \right) \right) \\ & - \frac{\partial P}{\partial \eta} - 5 \left(\frac{a'}{a} + \Phi' \right) P + \frac{\partial \Psi}{\partial x^i} v_b^i \rho_b = \Delta E_{\text{Compton}}, \end{aligned} \quad (3.14)$$

where

$$h \equiv \frac{5P}{2\rho_b} = \frac{5 \sum_{(\alpha)} S n_{(\alpha)} T_b}{2 \sum_{(\alpha)} m_{(\alpha)} S n_{(\alpha)}}, \quad (3.15)$$

and

$$\Delta E_{\text{Compton}} \equiv \frac{1}{2m_e} \frac{a}{a_0} \int \frac{d^3 \mathbf{q}_{(e)}}{(2\pi)^3} \mathbf{q}_{(e)}^2 C[f_{(e)}]_{\text{Compton}}. \quad (3.16)$$

IV. COLLISION TERM

Here let us evaluate the collision term between the electron and the photon. The explicit form of $C[f_{(e)}]_{\text{Compton}}$ is

$$\begin{aligned} C[f_{(e)}(\mathbf{q})]_{\text{Compton}} = & \frac{1}{2q^0} \int \int \int \frac{2d^3 \mathbf{p}}{(2\pi)^3 2p^0} \frac{2d^3 \mathbf{p}'}{(2\pi)^3 2p'^0} \frac{2d^3 \mathbf{q}'}{(2\pi)^3 2q'^0} (2\pi)^4 \delta^{(4)}(p + q - p' - q') |M|^2 \\ & \times \left\{ (1 + f_{(\gamma)}(\mathbf{p})) f_{(\gamma)}(\mathbf{p}') f_{(e)}(\mathbf{q}') - (1 + f_{(\gamma)}(\mathbf{p}')) f_{(\gamma)}(\mathbf{p}) f_{(e)}(\mathbf{q}) \right\}, \end{aligned} \quad (4.1)$$

where $f_{(\gamma)}(\mathbf{p})$ is the photon distribution function, \mathbf{p} and \mathbf{p}' are the photon momenta, $p^0 \equiv |\mathbf{p}|$, $p'^0 \equiv |\mathbf{p}'|$ and $\delta^{(4)}(p + q - p' - q') = \delta(p^0 + q^0 - p'^0 - q'^0) \delta^{(3)}(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}')$ is the 4-dimensional Dirac's delta

function. $|M|^2$ is the matrix element summing and averaging over the electron spin and the photon polarization [9] which is described as,

$$|M|^2 = \frac{(4\pi)^2}{2} \alpha_{\text{EM}}^2 \left(\frac{\tilde{p}'}{\tilde{p}} + \frac{\tilde{p}}{\tilde{p}'} - \sin^2 \tilde{\beta} \right). \quad (4.2)$$

Here \tilde{p} and $\tilde{\beta}$ are the photon energy and the photon scattering angle, respectively, in the electron rest frame, and $\alpha_{\text{EM}}^2 = 3m_e^2 \sigma_T / 8\pi$.

Since the explicit integration to get the collision term is very tedious, instead of performing the integration of (4.1), we here consider the Boltzmann equation for the photon distribution function

$$\frac{\partial f_{(\gamma)}}{\partial t} + \frac{\partial f_{(\gamma)}}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f_{(\gamma)}}{\partial p^i} \frac{dp^i}{dt} = C[f_{(\gamma)}]_{\text{Compton}}, \quad (4.3)$$

where

$$\begin{aligned} C[f_{(\gamma)}(\mathbf{p})]_{\text{Compton}} = & \frac{1}{2p^0} \int \int \int \frac{2d^3\mathbf{q}}{(2\pi)^3 2q^0} \frac{2d^3\mathbf{q}'}{(2\pi)^3 2q'^0} \frac{2d^3\mathbf{p}'}{(2\pi)^3 2p'^0} (2\pi)^4 \delta^{(4)}(p + q - p' - q') |M|^2 \\ & \times \frac{1}{2} \left\{ (1 + f_{(\gamma)}(\mathbf{p})) f_{(\gamma)}(\mathbf{p}') f_{(e)}(\mathbf{q}') - (1 + f_{(\gamma)}(\mathbf{p}')) f_{(\gamma)}(\mathbf{p}) f_{(e)}(\mathbf{q}) \right\}. \end{aligned} \quad (4.4)$$

Note the factor of 1/2 in the last line. We should put this factor because we have defined the electron distribution function by Eq.(3.1) or Eq.(4.5), where the spin degree of freedom is summed.

The integration of the collision term Eq.(4.4) has been performed by Dodelson & Jubas [8] and Hu, Scott & Silk [7]. The calculation is based on the assumption that the electron motion is non-relativistic, and is performed by expanding in terms of $O(1/m_e)$. In particular, in the paper by Dodelson & Jubas [8], the integration of the collision term is carried by assuming the same form of the electron distribution function as Eq.(3.1), i.e.,

$$f_{(e)} = n_e(\mathbf{x}) \left(\frac{2\pi}{m_e T_b(\mathbf{x})} \right)^{3/2} \exp \left[-\frac{(\mathbf{q}_e - m_e \mathbf{v}_b(\mathbf{x}))^2}{2m_e T_b(\mathbf{x})} \right], \quad (4.5)$$

and by expressing the photon distribution function in the power expansion of $O(1/m_e)$ up to the second order

$$f_{(\gamma)} = f_{(\gamma)}^{(0)}(p^0) + f_{(\gamma)}^{(1)}(\mathbf{p}) + f_{(\gamma)}^{(2)}(\mathbf{p}). \quad (4.6)$$

Following their result [8] *, the collision term is written in the second order of perturbation as

$$C[f_{(\gamma)}(\mathbf{p})]_{\text{Compton}} = C^{(1)} + C^{(2)}, \quad (4.7)$$

with

$$C^{(1)} = n_e \sigma_T \left[f_{(\gamma)0}^{(1)} + \frac{1}{2} f_{(\gamma)2}^{(1)} P_2(\mu) - f_{(\gamma)}^{(1)} - p^0 \frac{\partial f_{(\gamma)}^{(0)}}{\partial p^0} \mu v_b \right], \quad (4.8)$$

and

*Eq.(5.1) in their paper seems to contain a typographical error.

$$\begin{aligned}
C^{(2)} = n_e \sigma_T & \left[f_{(\gamma)0}^{(2)} + \frac{1}{2} f_{(\gamma)2}^{(2)} P_2(\mu) - f_{(\gamma)}^{(2)} \right. \\
& + v_b^2 p^0 \frac{\partial f_{(\gamma)}^{(0)}}{\partial p^0} (\mu^2 + 1) + v_b^2 (p^0)^2 \frac{\partial^2 f_{(\gamma)}^{(0)}}{(\partial p^0)^2} \left(\frac{11}{20} \mu^2 + \frac{3}{20} \right) \\
& \left. + \frac{1}{m_e (p^0)^2} \frac{\partial}{\partial p^0} \left\{ (p^0)^4 \left(T_b \frac{\partial f_{(\gamma)}^{(0)}}{\partial p^0} + f_{(\gamma)}^{(0)} (1 + f_{(\gamma)}^{(0)}) \right) \right\} \right], \tag{4.9}
\end{aligned}$$

where

$$f_{(\gamma)l}^{(i)} = \int_{-1}^1 \frac{d\mu}{2} P_l(\mu) f_{(\gamma)}^{(i)}, \tag{4.10}$$

$$\mu = \frac{\mathbf{v}_b \cdot \mathbf{p}}{|\mathbf{v}_b| |\mathbf{p}|}, \tag{4.11}$$

with $P_l(\mu)$ being a Legendre polynomial.

The energy transfer rate is described as

$$\Delta E_{\text{Compton}} = -\frac{a}{a_0} 2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} p^0 C[f_{(\gamma)}(\mathbf{p})]_{\text{Compton}}. \tag{4.12}$$

Therefore we obtain

$$\Delta E_{\text{Compton}} = -\frac{a}{a_0} \frac{4n_e \sigma_T \rho_{\gamma}^{(0)}}{m_e} \left(\frac{m_e}{3} \mathbf{v}_b^2(\mathbf{x}) - T_{\gamma}^{(0)} + T_b(\mathbf{x}) \right), \tag{4.13}$$

where $T_{\gamma}^{(0)}$ is the background photon temperature and $\rho_{\gamma}^{(0)} = \pi^2 T_{\gamma}^{(0)4}/15$ is the background photon energy density.

The momentum transfer rate can be found from

$$\Delta V_{\text{Compton}} = -\frac{a}{a_0} 2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \mathbf{p} C[f_{(\gamma)}(\mathbf{p})]_{\text{Compton}}, \tag{4.14}$$

and the result of the integration in the leading order of perturbations gives the well known form [5],

$$\Delta V_{\text{Compton}} = \frac{4}{3} \frac{a}{a_0} n_e \sigma_T \rho_{\gamma}^{(0)} (\mathbf{v}_{\gamma} - \mathbf{v}_b). \tag{4.15}$$

V. PERTURBATION EQUATIONS

Electrons(e), neutral and ionized hydrogen atoms(H) and helium atoms(He) are the particle species of the baryon-electron fluid. The number densities for each species are written

$$n_e = x \left(1 - \frac{y_p}{2} \right) n_B, \tag{5.1}$$

$$n_H = \left(1 - y_p \right) n_B, \tag{5.2}$$

$$n_{\text{He}} = \frac{y_p}{4} n_B, \tag{5.3}$$

where y_p is the primordial helium mass fraction, n_H and n_{He} are the number density of neutral and ionized hydrogen and helium, respectively, $n_B \equiv n_H + 4n_{\text{He}}$ is the total baryon number density and x is an ionization fraction defined as $x \equiv n_e / (n_H + 2n_{\text{He}})$. Then,

$$\rho_b = \sum_{(\alpha)} m_{(\alpha)} S n_{(\alpha)} \simeq m_p n_B \quad (5.4)$$

$$P = \sum_{(\alpha)} S n_{(\alpha)} T_b = \left(1 + x - \frac{y_p}{2} \left(x + \frac{3}{2}\right)\right) n_B T_b, \quad (5.5)$$

where m_p is the proton mass.

First let us summarize the equations for the baryon-electron fluid in the expanding universe. The continuity equation is (3.7). The Euler equation (3.10) reduces to

$$\frac{\partial v_b^i}{\partial \eta} + \left(\frac{a'}{a} + \Phi'\right) v_b^i + v_b^j \frac{\partial v_b^i}{\partial x^j} + \frac{1}{\rho_b} \frac{\partial P}{\partial x^i} + \frac{\partial \Psi}{\partial x^i} = \frac{a}{a_0} \frac{n_e \sigma_T}{R} \left(v_\gamma^i - v_b^i\right), \quad (5.6)$$

with the use of Eq.(3.7) and $R \equiv 3\rho_b/4\rho_\gamma^{(0)}$. The energy equation (3.14) reduces to

$$\begin{aligned} \frac{\partial}{\partial \eta} \left(\frac{\mathbf{v}_b^2}{2} + h \right) + 2 \left(\frac{a'}{a} + \Phi' \right) \left(\frac{\mathbf{v}_b^2}{2} + h \right) + v_b^i \frac{\partial}{\partial x^i} \left(\frac{\mathbf{v}_b^2}{2} + h \right) - \frac{1}{\rho_b} \frac{\partial P}{\partial \eta} - \frac{5}{\rho_b} \left(\frac{a'}{a} + \Phi' \right) P \\ + \frac{\partial \Psi}{\partial x^i} v_b^i = 4 \frac{a}{a_0} \frac{x(1 - y_p/2) \sigma_T \rho_\gamma^{(0)}}{m_e m_p} \left(T_\gamma^{(0)} - T_b(\mathbf{x}) - \frac{m_e}{3} \mathbf{v}_b^2 \right), \end{aligned} \quad (5.7)$$

by using Eqs.(3.7), (5.1) and (5.4). We supplement the equation of state,

$$h = \frac{5P}{2\rho_b}, \quad P = \frac{\rho_b}{m_p} \left(1 + x - \frac{y_p}{2} \left(x + \frac{3}{2} \right) \right) T_b. \quad (5.8)$$

Now we solve the above equations by a perturbative method assuming a small deviations from the uniformity. Define the perturbative expansions as follows,

$$\rho_b = \rho_b^{(0)}(\eta)(1 + \delta_b(\eta, \mathbf{x})), \quad (5.9)$$

$$h = h^{(0)}(\eta)(1 + \Delta_h(\eta, \mathbf{x})), \quad (5.10)$$

$$P = P^{(0)}(\eta)(1 + \Delta_P(\eta, \mathbf{x})), \quad (5.11)$$

$$T_b = T_b^{(0)}(\eta)(1 + \Delta_{T_b}(\eta, \mathbf{x})), \quad (5.12)$$

$$x = x^{(0)}(\eta) + \delta x(\eta, \mathbf{x}), \quad (5.13)$$

together with $\mathbf{v}_b = \mathbf{v}_b(\eta, \mathbf{x})$.

The zero-th order equations of (3.7) and (5.7) are

$$\rho_b^{(0)'} + 3 \frac{a'}{a} \rho_b^{(0)} = 0, \quad (5.14)$$

and

$$\frac{\partial h^{(0)}}{\partial \eta} + 2 \frac{a'}{a} h^{(0)} - \frac{1}{\rho_b^{(0)}} \left(\frac{\partial P^{(0)}}{\partial \eta} + 5 \frac{a'}{a} P^{(0)} \right) = 4 \frac{a}{a_0} \frac{x^{(0)}(1 - y_p/2) \sigma_T \rho_\gamma^{(0)}}{m_e m_p} \left(T_\gamma^{(0)} - T_b^{(0)} \right), \quad (5.15)$$

respectively. Eq.(5.15) reduces to [10]

$$T_b^{(0)'} + 2 \frac{a'}{a} T_b^{(0)} = \eta_E^{-1} (T_\gamma^{(0)} - T_b^{(0)}), \quad (5.16)$$

where we have neglected the term proportional to $x^{(0)'}T_b^{(0)}$ [†] and used the zero-th order equation of state,

$$h^{(0)} = \frac{5P^{(0)}}{2\rho_b^{(0)}}, \quad P^{(0)} = \frac{\rho_b^{(0)}}{m_p} \left(1 + x^{(0)} - \frac{y_p}{2} \left(x^{(0)} + \frac{3}{2} \right) \right) T_b^{(0)}. \quad (5.17)$$

We defined the Compton energy transfer time scale η_E as

$$\eta_E^{-1} = \frac{8}{3} \frac{a}{a_0} \frac{x^{(0)} \left(1 - y_p/2 \right) \sigma_T \rho_\gamma^{(0)}}{m_e \left(1 + x^{(0)} - (x^{(0)} + 3/2)y_p/2 \right)}. \quad (5.18)$$

The solution of Eq.(5.16) is discussed in appendix C.

The perturbed equations of (3.7) and (5.6) are

$$\delta_b' + 3\Phi' + \frac{\partial v_b^i}{\partial x^i} = 0, \quad (5.19)$$

and

$$v_b^{i'} + \frac{a'}{a} v_b^i + \frac{P^{(0)}}{\rho_b^{(0)}} \frac{\partial \Delta_P}{\partial x^i} + \frac{\partial \Psi}{\partial x^i} = \frac{a}{a_0} \frac{n_e \sigma_T}{R} \left(v_\gamma^i - v_b^i \right). \quad (5.20)$$

The perturbed equation of (5.7) yields

$$\begin{aligned} \Delta_h' + \left(\frac{5}{3} \frac{h^{(0)'}}{h^{(0)}} - \frac{2}{3} \frac{P^{(0)'}}{P^{(0)}} \right) \Delta_h - \frac{2}{3} \delta_b' \\ = \eta_E^{-1} \left(-\Delta_h + \frac{\delta x}{x^{(0)}} \frac{T_\gamma^{(0)} - T_b^{(0)}}{T_b^{(0)}} + \frac{\left(1 - y_p/2 \right) \delta x}{1 + x^{(0)} - (x^{(0)} + 3/2)y_p/2} \right), \end{aligned} \quad (5.21)$$

where we have used the perturbative part of the equation of state as follows,

$$\Delta_h = \Delta_P - \delta_b, \quad (5.22)$$

and

$$\Delta_P = \delta_b + \Delta_{T_b} + \frac{\left(1 - y_p/2 \right) \delta x}{1 + x^{(0)} - (x^{(0)} + 3/2)y_p/2}. \quad (5.23)$$

In order to complete the perturbation equations, we need an equation to specify the evolution of the perturbation of the ionization rate δx in equations (5.21) and (5.23), which we have considered in the appendix A. In the early phase of the recombination the terms proportional to δx become important.

[†] As we will see in the below, the matter temperature follows the photon temperature in the recombination regime. This is because the energy transfer time scale η_E is small enough. After recombination, the fraction of the residual ionization $x^{(0)}$ is almost fixed, and the time variation of $x^{(0)}$ is small. This is the reason why we neglected the term proportional to $x^{(0)'}T_b^{(0)}$.

VI. SOUND WAVES AFTER THE RECOMBINATION

As an application of the basic relations obtained in the previous sections, we consider the behavior of the sound wave of the baryon-electron fluid after the recombination epoch. Neglecting a perturbation of the gravitational potential, we get basic equations from Eqs.(5.19), (5.20), (5.21) as

$$\delta_b' + \frac{\partial v_b^i}{\partial x^i} = 0, \quad (6.1)$$

$$v_b^{i'} + \frac{a'}{a} v_b^i + \frac{P^{(0)}}{\rho_b^{(0)}} \frac{\partial \Delta_P}{\partial x^i} = 0, \quad (6.2)$$

and

$$\Delta_h' + \left(\frac{5}{3} \frac{h^{(0)'}}{h^{(0)}} - \frac{2}{3} \frac{P^{(0)'}}{P^{(0)}} \right) \Delta_h - \frac{2}{3} \delta_b' = -\eta_E^{-1} \Delta_h. \quad (6.3)$$

Here we have also neglected a perturbation of the ionization fraction and an interaction with the photon in the Euler equation because the momentum transfer rate is so small after the decoupling. The wave equation is derived from Eqs.(6.1) and (6.2), together with Eq.(5.22), as

$$\delta_b'' + \frac{a'}{a} \delta_b' - \frac{P^{(0)}}{\rho_b^{(0)}} \frac{\partial^2}{\partial \mathbf{x}^2} (\Delta_h + \delta_b) = 0. \quad (6.4)$$

This must be coupled with the energy equation (6.3) in order to investigate the sound oscillation of the baryon-electron fluid.

Now let us omit the time-dependence of the background quantities. When the time scale of the sound oscillation is shorter than the Hubble time, this is a good approximation. Then Eqs.(6.4) and (6.3) reduce to simple equations

$$\delta_b(k, \eta)'' + \frac{P^{(0)}}{\rho_b^{(0)}} k^2 (\Delta_h(k, \eta) + \delta_b(k, \eta)) = 0, \quad (6.5)$$

$$\Delta_h(k, \eta)' = \frac{2}{3} \delta_b(k, \eta)' - \eta_E^{-1} \Delta_h(k, \eta), \quad (6.6)$$

where we took the Fourier mode expansion by setting $\delta_b = \delta_b(k, \eta) e^{i\mathbf{k} \cdot \mathbf{x}}$, and $\Delta_h = \Delta_h(k, \eta) e^{i\mathbf{k} \cdot \mathbf{x}}$. The above equations yield

$$\eta_E \frac{d}{d\eta} \left(\frac{d^2 \delta_b(k, \eta)}{d\eta^2} + c_f^2 k^2 \delta_b(k, \eta) \right) + \left(\frac{d^2 \delta_b(k, \eta)}{d\eta^2} + c_e^2 k^2 \delta_b(k, \eta) \right) = 0. \quad (6.7)$$

Here we defined

$$c_f^2 = \frac{5}{3} \frac{P^{(0)}}{\rho_b^{(0)}}, \quad c_e^2 = \frac{P^{(0)}}{\rho_b^{(0)}}. \quad (6.8)$$

As is well known, c_f is the sound speed for the adiabatic state and c_e is the one for the isothermal state.

Taking the wave solution $\delta_b(k, \eta) \propto e^{-i\omega\eta}$, we get the following dispersion relation,

$$-i\eta_E \omega (\omega^2 - c_f^2 k^2) + \omega^2 - c_e^2 k^2 = 0. \quad (6.9)$$

In order to solve this equation, it is convenient to introduce the variable ϖ such as $\omega = i\varpi$ and we have

$$c_e k \eta_E \left(\frac{\varpi}{c_e k} \right) \left(\left(\frac{\varpi}{c_e k} \right)^2 + \frac{5}{3} \right) + \left(\frac{\varpi}{c_e k} \right)^2 + 1 = 0. \quad (6.10)$$

The sound speed c_s and the adiabatic index γ are defined as

$$c_s = -\text{Im}\left[\frac{\varpi}{k}\right], \quad \gamma = \frac{c_s^2}{c_e^2}. \quad (6.11)$$

Therefore we need to solve the cubic equation (6.10) in order to get the sound speed of the baryon-electron system.

FIG. 1. The behaviors of γ as a function of $1/(1+z)$. Each lines for the mass scales $M = 1M_\odot, 10^3M_\odot, 10^6M_\odot, 10^9M_\odot, 10^{12}M_\odot$, respectively, with the cosmological parameters, $h = 0.5, \Omega_0 = 1.0, \Omega_B = 0.1$. γ is changed in the earlier stage from 1 to 5/3 for the smaller scale perturbations.

A cubic equation can be solved exactly in an analytic form while the expression is complicated. It may be instructive to show the solution of Eq.(6.10) in a simple form with some approximation. Expanding the solution around the adiabatic state, i.e., expanding in terms of ϵ by setting the solution $\omega/k = c_f(1 + \epsilon)$, we find

$$\frac{\omega}{k} \simeq c_f \left(1 - \frac{1 + ikc_f\eta_E}{5(1 + (kc_f\eta_E)^2)} \right). \quad (6.12)$$

It is interesting to clarify the behavior of the adiabatic index γ in the expanding universe after the recombination, which we demonstrate as an example of the usefulness of our formalism in the below. As is clear from Eq.(6.9), the adiabatic index γ is ruled by the ratio of the Compton energy transfer time scale η_E to the sound oscillation time scale. If $\eta_E\omega \ll 1$, $\omega \simeq c_e k$ and $\gamma \simeq 1$. On the other hand, if $\eta_E\omega \gg 1$, $\omega \simeq c_f k$ and $\gamma \simeq 5/3$. Therefore the sound speed depends on scales of the perturbations, and it is expected that γ is changed from 1 to 5/3 as the universe expands.

Fig.1 shows γ as a function of $1/(1+z)$ by directly solving Eq.(6.10). The ionization fraction is calculated properly by solving the recombination process in the expanding universe with the cosmological

parameters in the figure caption [11,12]. As is expected, γ is changed from 1 to 5/3 in the earlier stage of the universe for smaller scale perturbations.

FIG. 2. The physical mass scales for the baryon perturbations. The cosmological parameters taken here are same as those in Fig.1 The definitions of these scales are summarized in appendix B. In the figure, ‘Horizon’ means the horizon scale, ‘Silk’ does the diffusion damping scale, ‘tight couple’ does the breaking scale of the tight coupling approximation of baryon and photon fluids, ‘Jeans’ does the Jeans scale.

To get a physical insight, we refer to the familiar illustrative figure, Fig.2, which gives temporal variations of various physical sizes in the expanding universe. The definitions of the curves are summarized in appendix B. (See also ref. [13].) The adiabatic index γ after the recombination is changed when the Compton energy transfer time is equal to the sound oscillation time scale. In this figure, we show the line on which the two time scales are equal, i.e., $\eta_E k c_e = 1$. Note that we use the (baryon) mass scale in the unit of the solar mass instead of k , with employing the relation $M = (4\pi\rho_b/3)(\pi a/ka_0)^3$. The mass and the comoving wave number which satisfy the relation $\eta_E k c_e = 1$ are expressed (see also appendix B),

$$k = \frac{9.2x^{(0)}(1 - y_p/2)}{(\mu^3 T_b^{(0)})^{1/2}} (1+z)^3 \text{ Mpc}^{-1}, \quad (6.13)$$

$$M = \frac{4.6 \times 10^{10} (\mu^3 T_b^{(0)})^{3/2}}{x^{(0)3} (1 - y_p/2)^3} (1+z)^{-9} \Omega_b h^2 M_\odot, \quad (6.14)$$

where we set the temperature of the microwave background at present $T_\gamma^{(0)}(t_0) = 2.726\text{K}$, T_b is the baryon temperature in unit of Kelvin, and $\mu = 1 - 3y_p/4$.

As is shown in Fig.2, the Jeans scale after the recombination has the plateau. In this stage the energy transfer between background photons and the baryon fluid is effective through the residual electrons, and the matter temperature follows the photon temperature. As the universe expands, however, the energy

transfer time rises above the Hubble expansion time. After that epoch the matter temperature cools adiabatically and drops as $T_b \propto 1/a^2$. The broken corner of the plateau is the critical time that the two time scales become equal, where this epoch is roughly estimated as $z \simeq 1000(\Omega_b h^2)^{2/5}$ [10] (see also appendix C).

FIG. 3. The behaviors of f_D as the function of $1/(1+z)$. Each lines are for the mass scales, $M = 1M_\odot, 10^3M_\odot, 10^6M_\odot, 10^9M_\odot, 10^{12}M_\odot$, respectively. The cosmological parameters are same as those in Fig.1 The damping phenomenon occurs when the γ is changed from 1 to 5/3.

The line of $\eta_E k c_e = 1$ crosses at the broken corner of the plateau of the Jeans mass scale. This necessarily happens because of the following reason. The Jeans scale is the scale at which the sound oscillation time is equal to the free fall time of the perturbation or the Hubble expansion time. Since the broken corner of the plateau is the epoch when the Hubble expansion time is equal to the energy transfer time through the Compton interaction between the background photons and the baryon-electron fluid, the cross point of two lines is the point when the sound oscillation time, the Hubble expansion time and the energy transfer time become all same.

Now let us discuss an interesting result derived from the dispersion relation (6.9). As it is apparent from the approximated solution, eq.(6.12), ω generally has an imaginary part. This imaginary part of ω represents the exponential damping of the wave oscillation if $\text{Im } \omega < 0$. Since the period of the sound oscillation is $T = 2\pi/|\text{Re } \omega|$, the damping factor for the amplitude during one period is

$$f_D = \exp \left[-2\pi \left| \frac{\text{Im } \omega}{\text{Re } \omega} \right| \right]. \quad (6.15)$$

We show the behavior of f_D in Fig.3, in the similar way to Fig.1 as the function of $1/(1+z)$. As is shown in Fig.3, this damping phenomenon may have an effect on the evolution of baryon perturbations. We should be notice that our discussions here are based on the assumption that the solutions have the wave form. In other word, this damping process is effective only inside the Jeans scale.

FIG. 4. The trajectory of the solution on $(\Delta_P - \delta_b)$ -plane. The horizontal axis is the baryon density perturbation $-\delta_b(k, \eta)$, and the vertical axis is the pressure perturbation $\Delta_P(k, \eta)$. We take $\eta_E k c_e = 0.1, 1, 10$ and 100 .

In order to understand this damping mechanism, we show another aspect of the solutions of Eq.(6.7). Introducing $\omega_R = \text{Re}\omega$, $\omega_I = \text{Im}\omega$, we take the solution

$$\delta_b(k, \eta) = -\cos(\omega_R \eta) e^{-\omega_I \eta}. \quad (6.16)$$

The initial value is $\delta_b(k, \eta = 0) = -1$. Here ω is obtained by solving the dispersion relation Eq.(6.9). For $\eta_E k c_e \gg 1$ and $\eta_E k c_e \ll 1$, solutions are mere harmonic oscillations with no damping. Therefore the solutions around $\eta_E k c_e \simeq 1$ should be investigated. By employing Eqs. (6.5) and (5.22), the trajectories of the solutions on $(\Delta_P - \delta_b)$ -plane are shown in Fig.4. The horizontal axis is $-\delta_b(k, \eta)$ and the vertical axis is $\Delta_P(k, \eta)$. The non-dimensional quantity $\eta_E k c_e$ is the unique physical parameter of the equation. We have chosen $\eta_E k c_e = 0.1, 1, 10$, and 100 in Fig.4.

From the relation $-\delta_b = \delta V/V$, where V is the volume for unit particle number, the horizontal axis can be regarded as the change of volume per unit particle number. Thus we can regard the trajectories in Fig.4 in the similar way to the thermodynamical cycles in (pressure-volume)-plane. The damping of the oscillation is in proportion to the deviation in one cycle. The most significant damping occurs when $\eta_E k c_e = 1$ (Fig.4(b)). The damping does not happen if the deviation in one cycle is negligible as we can see in Fig.4(d).

VII. SUMMARY AND DISCUSSIONS

We have formulated equations for a fluid system with the electron, neutral and ionized hydrogen atoms and neutral helium atoms with taking into account the energy transfer between the background photons and the residual electrons through the Compton interaction. Using this formulation, we have studied the time evolution of the sound speed and the Jeans scale after the recombination. We found that the behavior of the adiabatic index γ after the recombination is controlled by the ratio of the Compton energy transfer time scale to the sound oscillation time scale. Then γ (or the sound speed) depends on scales of the perturbations, and is changed from 1 to 5/3 in earlier stage for smaller scale perturbations. This formalism enables us to calculate the linear evolution of the very small scale baryon density perturbations [13].

We have also discussed the small damping feature of the sound oscillation when γ changes from 1 to 5/3. This effect works inside the Jeans scale and seems to be negligible on scales $M \gtrsim 10^6 M_\odot$. If early reionization occurs, however, the ionization fraction and the Jeans scale increase. And γ changes from 5/3 to 1. The damping of the baryon perturbation would be effective on larger scales during the process of the reionization of the universe. It has been pointed out that the neutral gas cloud could have an instability in a specific photon background [14]. Since we have not consider the specific situation in our paper, an unstable mode does not appear in our equations. In the present paper we have also neglected perturbations to the ionization fraction. Future work may be required on this point.

It will be important to quantify the limitation of our formalism. Our basic assumption is the fact that the baryon-electron system is treated as tightly coupled single-fluid. This assumption becomes not being correct when the collision time scale of neutral interaction rises above the Hubble expansion time. This epoch can be estimated in the following way [15]. The matter temperature can be written as $T_b = 4.5 \times 10^{-3} (\Omega_b h^2)^{-2/5} (1+z)^2$ [K], after the energy transfer through the Compton interaction becomes ineffective (see appendix C). Then the mean free time for collision between neutral hydrogen atoms is $t_c \simeq 1/(n_B \sigma v)$, where σ is the cross section of neutral atoms $\sigma \simeq \pi r_B^2$ (r_B is the Bohr radius), and $v = \sqrt{3T_b/m_H}$. The ratio of t_c to the Hubble expansion time is therefore expressed as $t_c H \simeq 3.1 (\Omega_0 h^2)^{1/2} (\Omega_b h^2)^{-4/5} (1+z)^{-5/2}$. Eventually we get the red-shift at $t_c H = 1$ as $(1+z) \simeq 1.6 (\Omega_0 h^2)^{0.2} (\Omega_b h^2)^{-0.32}$. This is small enough as long as we consider the linear stage of the density perturbations.

Numerical simulations and semi-analytic calculations of structure formation employ the linear matter power spectrum as their initial conditions. This linear matter spectrum is usually calculated without taking into account the baryon pressure term after the recombination. It is appropriate, however, only if the structures which are larger than the Jeans scale are considered. According to the hierarchical clustering scenario, smaller objects are formed earlier than larger ones. We expect the scale of the first collapsing object is very close to the Jeans scale. Therefore very accurate estimate of the matter spectrum including the baryon pressure term is necessary to understand the early formation of bound objects [13]. Once the first collapsing objects are formed, they may cool down through formation of hydrogen molecules and may fragment into smaller radiating objects like stars and/or quasars [3, 16–18]. The first radiating objects formed Strömgren sphere around them [19] and may eventually ionize all the surrounding gas by UV radiation. This reionization process changes the Jeans scale [20]. After that, full numerical or semi-analytic calculations including collapsing objects are required.

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APPENDIX A: RATE EQUATION

In order to complete the fluid equations we should add the rate equation. For simplicity, we neglect the helium fraction, i.e., $n_B = n_H$, and neglect the helium recombination process in the rate equation. Then, the rate equation is [10]

$$\frac{\partial x}{\partial \eta} = -\langle \sigma v \rangle \frac{a}{a_0} n_B C \left(x^2 - (1-x) \frac{x_{\text{eq}}^2}{1-x_{\text{eq}}} \right), \quad (\text{A1})$$

where $\langle \sigma v \rangle$ is the rate coefficient for recombination to excited states, x_{eq} is the equilibrium ionization fraction which is given by the Saha equation

$$\frac{x_{\text{eq}}^2}{1-x_{\text{eq}}} = \frac{m_p}{\rho_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-13.6\text{eV}/T_b}, \quad (\text{A2})$$

and

$$C = \frac{1 + K \Lambda n_{1s}}{1 + K(\Lambda + \beta_e) n_{1s}}. \quad (\text{A3})$$

Here n_{1s} is the number density of hydrogen in the electron ground state, for which we approximate $n_{1s} = n_B(1-x)$, Λ is the decay rate from the excited state, $\beta_e = \langle \sigma v \rangle (m_e T_b / 2\pi)^{3/2} e^{-3.4 \text{ eV}/T_e}$ and $K = (a/\dot{a}) \lambda_\alpha^3 / 8\pi$ with λ_α being the Lyman alpha photon wave length.

The Zero-th order equations of the rate equation and the Saha equation are the same forms of Eqs.(A1) and (A2) replaced the variables with the zero-th order quantities.

Next we consider the perturbation of the rate equation. We define

$$x_{\text{eq}} = x_{\text{eq}}^{(0)}(\eta) + \delta x_{\text{eq}}(\eta, \mathbf{x}), \quad (\text{A4})$$

then

$$\begin{aligned} \frac{d\delta x}{d\eta} = \frac{a}{a_0} \langle \sigma v \rangle n_B^{(0)} C \left[\left(x^{(0)2} - \frac{(1-x^{(0)})x_{\text{eq}}^{(0)2}}{1-x_{\text{eq}}^{(0)}} \right) \left(\frac{\delta C}{C} + \delta_b \right) \right. \\ \left. + \left(2x^{(0)} + \frac{x_{\text{eq}}^{(0)2}}{1-x_{\text{eq}}^{(0)}} \right) \delta x - \frac{(1-x^{(0)})x_{\text{eq}}^{(0)}(2-x_{\text{eq}}^{(0)})}{(1-x_{\text{eq}}^{(0)})^2} \delta x_{\text{eq}} \right], \end{aligned} \quad (\text{A5})$$

where

$$\frac{\delta C}{C} = \frac{-K\beta_e n_B^{(0)} \left((1-x^{(0)})\delta_b - \delta x \right)}{\left(1 + K\Lambda n_B^{(0)}(1-x^{(0)}) \right) \left(1 + K(\Lambda + \beta_e) n_B^{(0)}(1-x^{(0)}) \right)}. \quad (\text{A6})$$

We have assumed that $\langle \sigma v \rangle$ and β_e are constant.

The perturbation of the Saha equation can be written as

$$\delta x_{\text{eq}} = \frac{\partial x_{\text{eq}}(P^{(0)}, \rho_b^{(0)})}{\partial \rho_b^{(0)}} \Big|_{P^{(0)}} \rho_b^{(0)} \delta_b + \frac{\partial x_{\text{eq}}(P^{(0)}, \rho_b^{(0)})}{\partial P^{(0)}} \Big|_{\rho_b^{(0)}} P^{(0)} \Delta_P. \quad (\text{A7})$$

APPENDIX B: PHYSICAL SCALES

In this appendix, we summarize the physical scales which are important for the evolution of the baryon perturbations on small scales. The definitions of the physical scales in Fig.2 are given. Mass scales are defined by the amount of baryonic components inside the systems. Here we have set that the temperature of the microwave background at present $T_\gamma(t_0) = 2.726K$. We write f_ν as the neutrino fraction of the energy density in the massless particles, and $f_\nu = 0.405$ in case of the standard three families of massless neutrinos.

First of all, we use the notation for the physical wave number (length) and the comoving wave number (length) as

$$k^{\text{comv}} = \left(\frac{a}{a_0} \right) k^{\text{phys}}, \quad \lambda^{\text{comv}} = \left(\frac{a_0}{a} \right) \lambda^{\text{phys}} = 2\pi/k^{\text{comv}}. \quad (\text{B1})$$

It will be useful to give the relation between the red-shift z and the scale factor normalized at the matter radiation equality a_{eq} ,

$$\frac{a}{a_{\text{eq}}} = 4.04 \times 10^4 (1 - f_\nu) \Omega_0 h^2 (1 + z)^{-1}. \quad (\text{B2})$$

1. Horizon Scale

We define the horizon wave number and the horizon mass as

$$\frac{1}{k_H^{\text{comv}}} = \eta, \quad (\text{B3})$$

$$M_H = \frac{4\pi\rho_b}{3} \left(\frac{\lambda_H^{\text{phys}}}{2} \right)^3 = \frac{4\pi\rho_b}{3} \left(\frac{\pi}{k_H^{\text{phys}}} \right)^3, \quad (\text{B4})$$

which derive

$$k_{\text{H}}^{\text{comv}} = 3.35 \times 10^{-2} \left(\sqrt{1 + \frac{a}{a_{\text{eq}}}} - 1 \right)^{-1} (1 - f_{\nu})^{1/2} \Omega_0 h^2 \text{ Mpc}^{-1}, \quad (\text{B5})$$

$$M_{\text{H}} = 9.57 \times 10^{17} \left(\sqrt{1 + \frac{a}{a_{\text{eq}}}} - 1 \right)^3 (1 - f_{\nu})^{-3/2} \Omega_{\text{b}} h^2 (\Omega_0 h^2)^{-3} M_{\odot}. \quad (\text{B6})$$

2. Jeans scale before the recombination

We define the Jeans wave length (wave number) and the Jeans mass as

$$\lambda_{\text{J}}^{\text{phys}} = \frac{2\pi}{k_{\text{J}}^{\text{phys}}} = \sqrt{\frac{\pi c_{\text{s}}^2}{G(\rho_{\text{m}} + \rho_{\gamma})}}, \quad (\text{B7})$$

$$M_{\text{J}} = \frac{4\pi\rho_{\text{b}}}{3} \left(\frac{\lambda_{\text{J}}^{\text{phys}}}{2} \right)^3 = \frac{4\pi\rho_{\text{b}}}{3} \left(\frac{\pi}{k_{\text{J}}^{\text{phys}}} \right)^3, \quad (\text{B8})$$

where

$$c_{\text{s}}^2 = \frac{1}{3(1+R)}, \quad (\text{B9})$$

and $\rho_{\text{m}} = \rho_{\text{b}} + \rho_{\text{dm}}$ with ρ_{dm} being the energy density of the dark component. Then we have

$$k_{\text{J}}^{\text{comv}} = 1.42 \times 10^{-1} \left(\frac{a_{\text{eq}}}{a} + (1 - f_{\nu}) \frac{a_{\text{eq}}^2}{a^2} \right)^{1/2} (1+R)^{1/2} (1 - f_{\nu})^{1/2} \Omega_0 h^2 \text{ Mpc}^{-1}, \quad (\text{B10})$$

$$M_{\text{J}} = 1.25 \times 10^{16} \left(\frac{a_{\text{eq}}}{a} + (1 - f_{\nu}) \frac{a_{\text{eq}}^2}{a^2} \right)^{-3/2} (1+R)^{-3/2} (1 - f_{\nu})^{-3/2} (\Omega_0 h^2)^{-3} \Omega_{\text{b}} h^2 M_{\odot}. \quad (\text{B11})$$

3. Jeans scale after the recombination

We can define the Jeans scale after the recombination as

$$\lambda_{\text{J}}^{\text{phys}} = \frac{2\pi}{k_{\text{J}}^{\text{phys}}} = \sqrt{\frac{\pi c_{\text{s}}^2}{G\rho_{\text{m}}}}, \quad (\text{B12})$$

with

$$c_{\text{s}}^2 = \gamma \frac{P^{(0)}}{\rho_{\text{b}}^{(0)}} = 9.18 \times 10^{-14} \gamma \mu T_{\text{b}}^{(0)}, \quad (\text{B13})$$

where $\mu = 1 - 3y_{\text{p}}/4$ for $x^{(0)} \ll 1$, and T_{b} is the matter temperature in unit of Kelvin. Then we have

$$k_{\text{J}}^{\text{comv}} = 2.71 \times 10^5 \left(\gamma \mu T_{\text{b}} \frac{a}{a_{\text{eq}}} \right)^{-1/2} (1 - f_{\nu})^{1/2} \Omega_0 h^2 \text{ Mpc}^{-1}, \quad (\text{B14})$$

$$M_{\text{J}} = 1.81 \times 10^{-3} \left(\gamma \mu T_{\text{b}} \frac{a}{a_{\text{eq}}} \right)^{3/2} (1 - f_{\nu})^{-3/2} (\Omega_0 h^2)^{-3} \Omega_{\text{b}} h^2 M_{\odot}. \quad (\text{B15})$$

4. Diffusion damping scale

We define the diffusion damping scales as [12]

$$\left(\frac{1}{k_D^{\text{comv}}}\right)^2 = \frac{1}{6} \int \frac{1}{\dot{\tau}} d\eta \frac{R^2 + 4(1+R)/5}{(1+R)^2}, \quad (\text{B16})$$

$$M_D = \frac{4\pi}{3} \rho_b \left(\frac{\pi}{k_D^{\text{phys}}}\right)^3, \quad (\text{B17})$$

where $\dot{\tau} = n_e \sigma_T (a/a_0)$. For $R \ll 1$, we get

$$k_D^{\text{comv}} = 1.38 \times 10^2 \left(\frac{u(a/a_{\text{eq}})}{x^{(0)}(1-y_p/2)}\right)^{-1/2} (1-f_\nu)^{5/4} (\Omega_0 h^2)^{3/2} (\Omega_b h^2)^{1/2} \text{ Mpc}^{-1}, \quad (\text{B18})$$

$$M_D = 1.38 \times 10^7 \left(\frac{u(a/a_{\text{eq}})}{x^{(0)}(1-y_p/2)}\right)^{3/2} (1-f_\nu)^{-15/4} (\Omega_0 h^2)^{-9/2} (\Omega_b h^2)^{-1/2} M_\odot, \quad (\text{B19})$$

where $u(y) = (\sqrt{1+y}(16-8y+6y^2)-16)/15$.

5. Breaking Scale of the tight coupling approximation

The breaking scale of the tight coupling approximation is defined by $1/k_{\text{BR}}^{\text{phys}} = 1/n_e \sigma_T$, i.e.,

$$k_{\text{BR}}^{\text{comv}} = n_e \sigma_T \frac{a}{a_0}, \quad (\text{B20})$$

$$M_{\text{BR}} = \frac{4\pi}{3} \rho_b \left(\frac{\pi}{k_{\text{BR}}^{\text{phys}}}\right)^3. \quad (\text{B21})$$

Then we have

$$k_{\text{BR}}^{\text{comv}} = 3.77 \times 10^4 \left(\frac{(a/a_{\text{eq}})^2}{x^{(0)}(1-y_p/2)}\right)^{-1} (1-f_\nu)^2 (\Omega_0 h^2)^2 (\Omega_b h^2) \text{ Mpc}^{-1}, \quad (\text{B22})$$

$$M_{\text{BR}} = 6.75 \times 10^{-1} \left(\frac{(a/a_{\text{eq}})^2}{x^{(0)}(1-y_p/2)}\right)^3 (1-f_\nu)^{-6} (\Omega_0 h^2)^{-6} (\Omega_b h^2)^{-2} M_\odot. \quad (\text{B23})$$

6. γ transition epoch after the recombination

The adiabatic index γ after the recombination is changed when the Compton energy transfer time is equal to the sound oscillation time scale. We define this scale by $1/k_c^{\text{comv}} = c_e \eta_E$, which leads

$$k_c^{\text{comv}} = \frac{6.08 \times 10^{14} x^{(0)} (1-y_p/2) (1-f_\nu)^3}{(\mu^3 T_e)^{1/2} (a/a_{\text{eq}})^3} (\Omega_0 h^2)^3 \text{ Mpc}^{-1}, \quad (\text{B24})$$

$$M_c = \frac{1.60 \times 10^{-31} (\mu^3 T_e)^{3/2} (a/a_{\text{eq}})^9}{x^{(0)3} (1-y_p/2)^3 (1-f_\nu)^9} \Omega_b h^2 (\Omega_0 h^2)^{-9} M_\odot, \quad (\text{B25})$$

where T_b is the matter temperature in unit of Kelvin as is mentioned above.

APPENDIX C: MATTER TEMPERATURE

As we have derived in §V, the matter temperature follows

$$T_b^{(0)'} + 2\frac{a'}{a}T_b^{(0)} = \eta_E^{-1}(T_\gamma^{(0)} - T_b^{(0)}), \quad (C1)$$

where $T_\gamma^{(0)}$ is the photon temperature, and the Compton energy transfer time scale η_E is defined as

$$\eta_E^{-1} = \frac{8}{3} \frac{a}{a_0} \frac{x^{(0)}(1 - y_p/2)\sigma_T\rho_\gamma^{(0)}}{m_e(1 + x^{(0)} - (x^{(0)} + 3/2)y_p/2)}. \quad (C2)$$

The formal solution is

$$T_b^{(0)} = \frac{1}{a^2} \int_0^\eta d\eta' a^2 T_\gamma^{(0)} \eta_E^{-1} \exp\left[-\int_{\eta'}^\eta d\eta'' \eta_E^{-1}\right]. \quad (C3)$$

We can obtain the epoch when the matter temperature deviates from the photon temperature as follows. This epoch is naturally defined as the epoch when the function $\eta_E^{-1} \exp(-\int_{\eta_0}^\eta d\eta' \eta_E^{-1})$ takes its maximum value. Thus we need to solve the equation $\dot{\eta}_E^{-1} + (\eta_E^{-1})^2 = 0$. From Eq. (C2),

$$\eta_E^{-1} = 1.8 \times 10^8 (1 - f_\nu)^3 (\Omega_0 h^2)^3 \frac{x^{(0)}(1 - y_p/2)}{(1 - 3y_p/4)} \left(\frac{a}{a_{\text{eq}}}\right)^{-3} \text{Mpc}^{-1}. \quad (C4)$$

Therefore we get the epoch as

$$\frac{a}{a_{\text{eq}}} = 2.3 \times 10^3 x^{(0)2/5} (\Omega_0 h^2)^{4/5}, \quad (C5)$$

with neglecting the time variation of the ionization fraction $x^{(0)}$. Here we set $y_p = 0.23$ and $f_\nu = 0.405$.

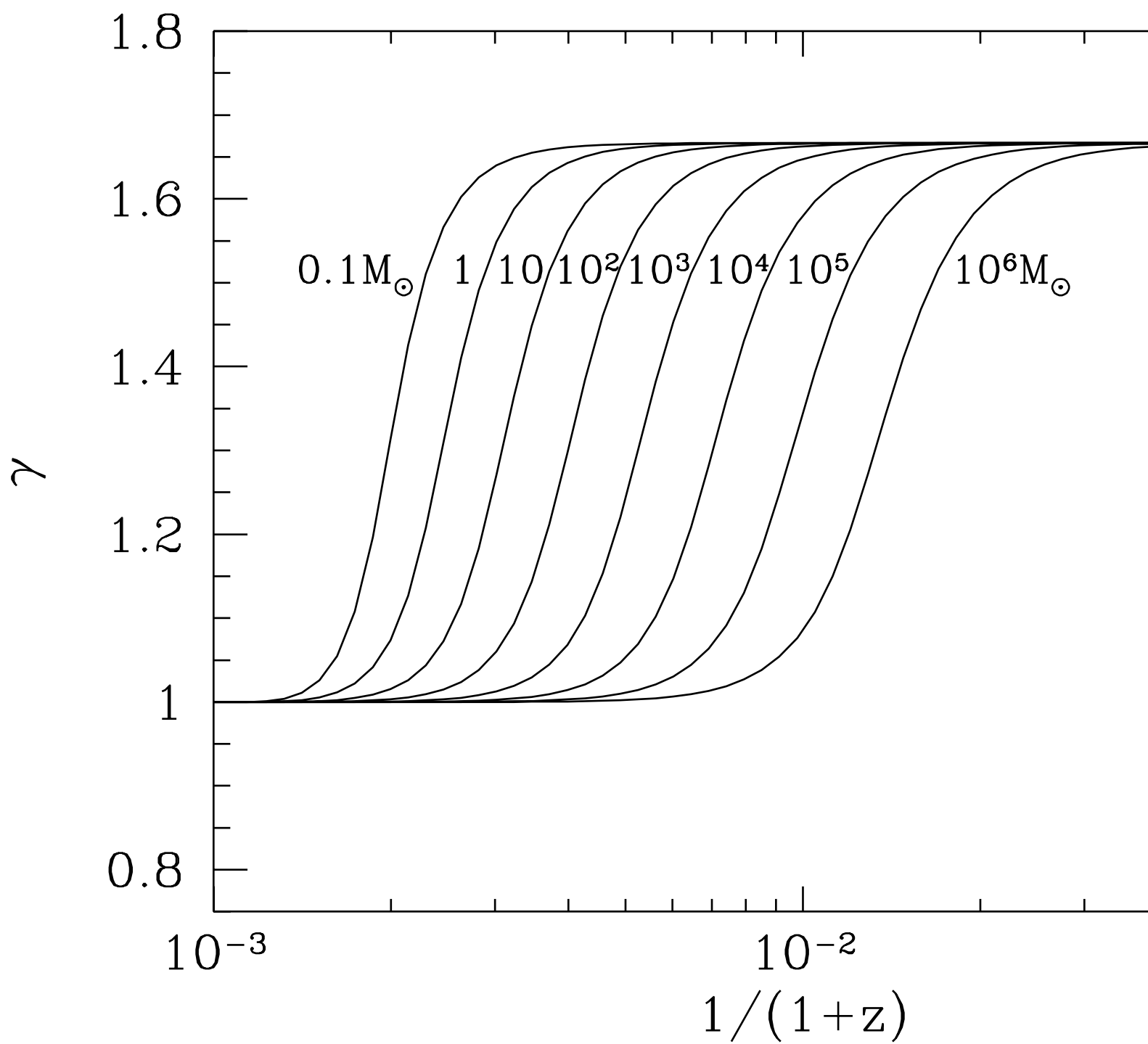
We approximate the fraction of the residual electron as [21]

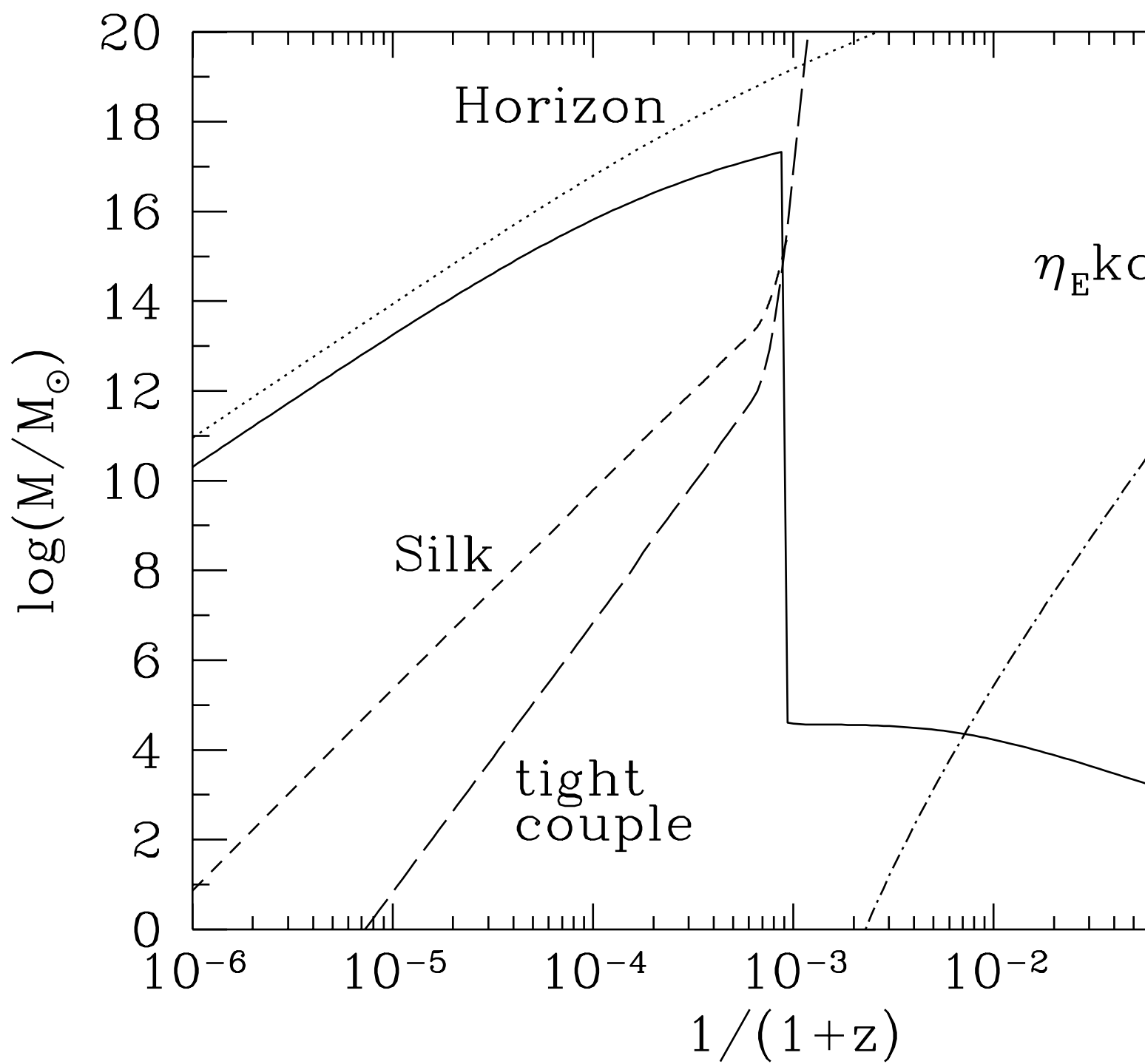
$$x^{(0)} \simeq 10^{-5} (\Omega_0 h^2)^{1/2} (\Omega_b h^2)^{-1}. \quad (C6)$$

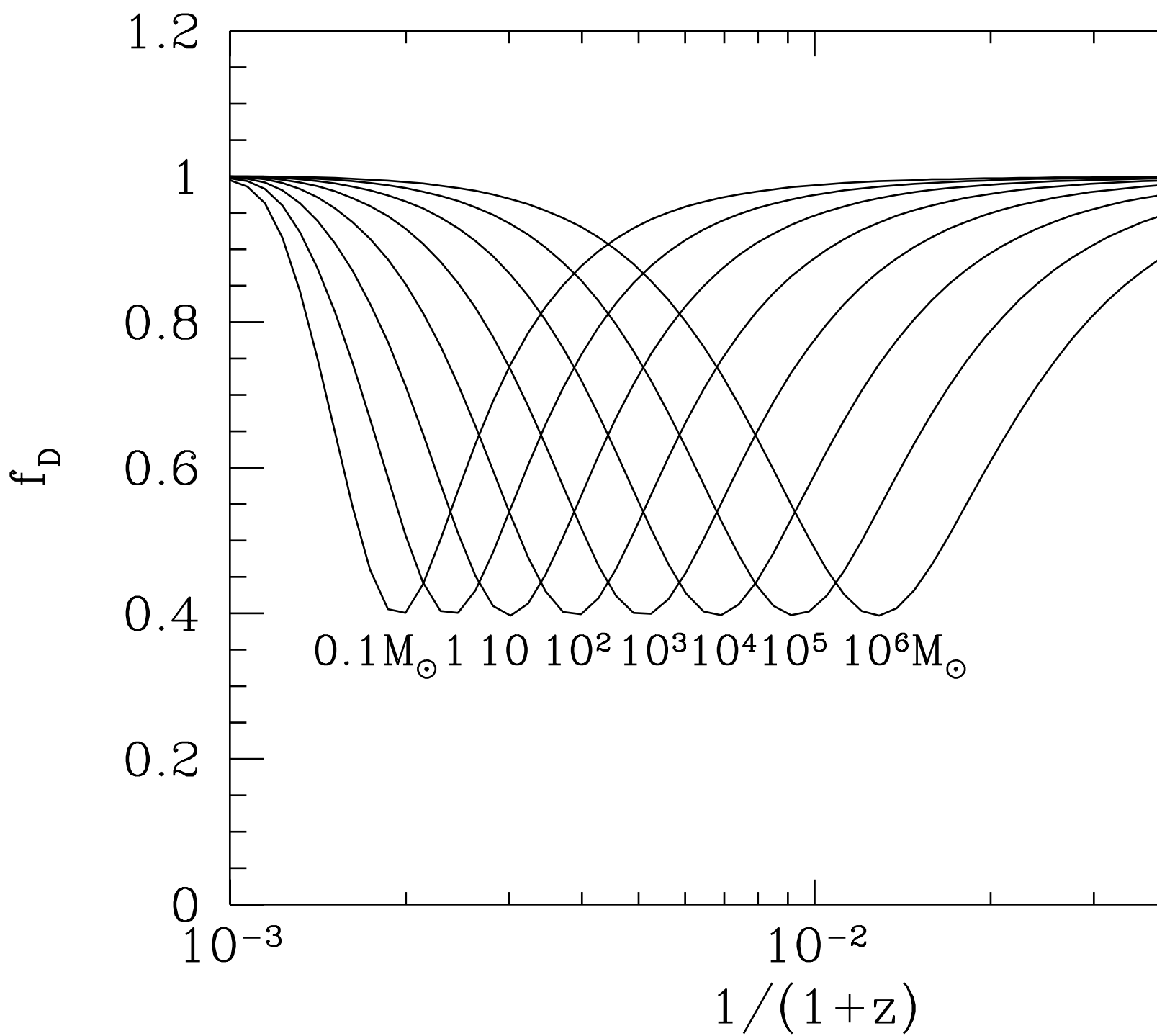
Then from Eq.(C5), we conclude that the matter temperature decouples from the photon's at $(1 + z) \simeq 1000(\Omega_b h^2)^{2/5}$. After this epoch, the matter temperature is adiabatically cooling.

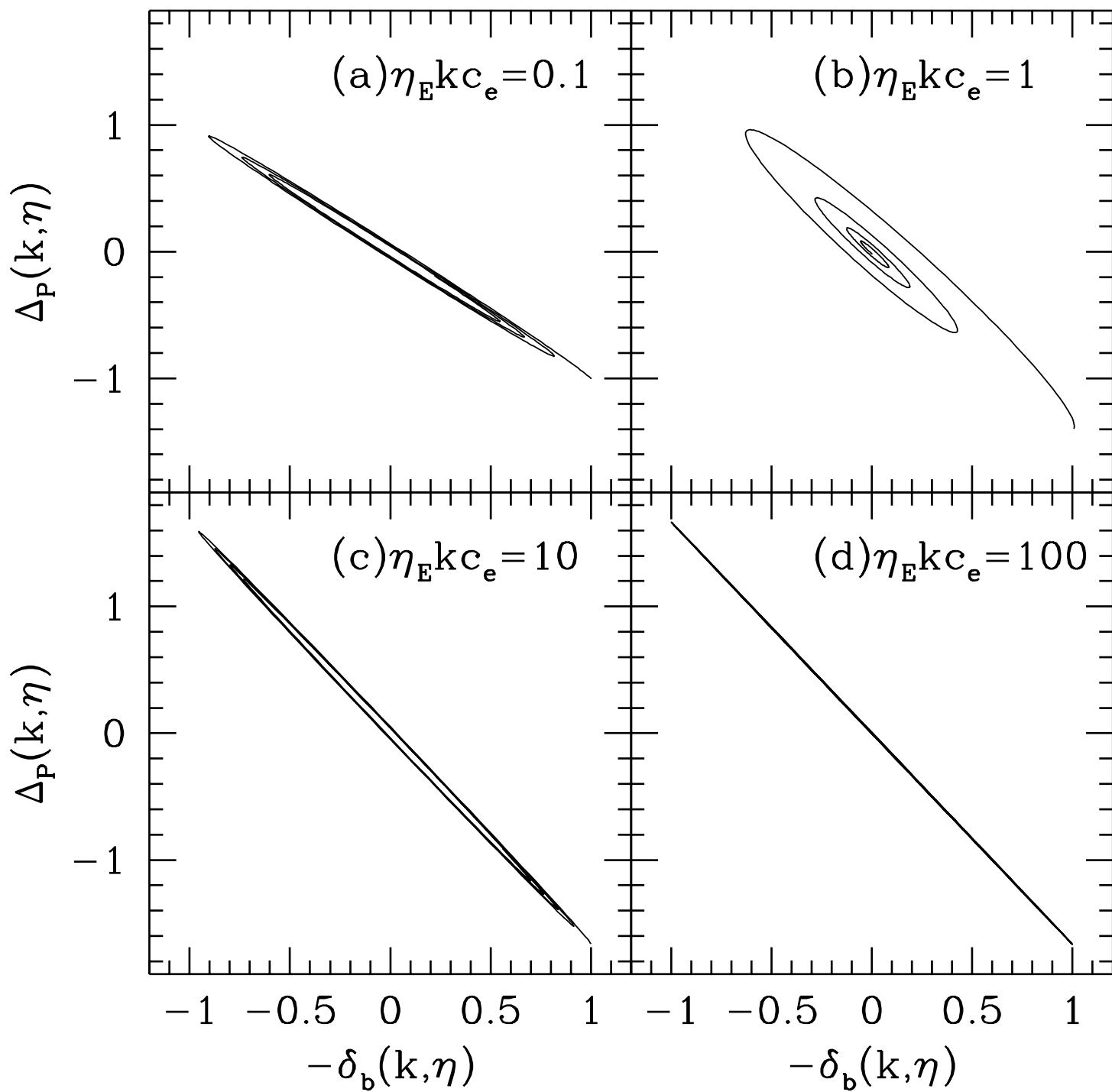
According to the fully numerical calculation [12], the matter temperature in this adiabatic cooling phase is well reproduced by the formula,

$$T_b^{(0)} = 4.5 \times 10^{-3} (1 + z)^2 (\Omega_b h^2)^{-2/5} \text{K}. \quad (C7)$$









Cosmological Baryon Sound Waves Coupled with the Primeval Radiation

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The fluid equations for the baryon-electron system in an expanding universe are derived from the Boltzmann equation. The effect of the Compton interaction is taken into account properly in order to evaluate the photon-electron collisional term. As an application, the acoustic motions of the baryon-electron system after recombination are investigated. The effective adiabatic index γ is computed for sound waves of various wavelengths, assuming the perturbation amplitude is small. The oscillations are found to be damped when γ changes from between 1 (for an isothermal process) to $5/3$ (for an adiabatic process).

98.80.-k, 51.10.+y, 52.35.Dm

I. INTRODUCTION

One of the biggest interest in astrophysics is to understand how structures in the universe are formed. Observations reveal that the universe is not simply homogeneous and isotropic but contains rich structures from stars to the large scale structure such as clusters of galaxies, super-clusters, the great wall and voids in it. Although the gravity itself does not have any particular scale, several physical scales associated with the structure formation, such as the Jeans scale, the Silk damping scale and the horizon scale at the matter-radiation equality epoch, are derived by the inclusion of an effect of primeval radiation [1–3]. Among them, the Jeans scale determines whether small density fluctuations can grow against pressure or not.

The linear evolution of small density perturbations has been well understood. Outside the Jeans scale, their evolution is described by the growing mode solution [4,5]. Assuming the spherical symmetry, we can follow the evolution of over-dense regions. The critical threshold of a density contrast to collapse is 1.69 in the Einstein-de Sitter universe and we can estimate the fraction of the collapsed mass by employing the extrapolation of the linear perturbation theory [6]. If the wave length of fluctuations is smaller than the Jeans scale, however, fluctuations cannot grow up to the critical threshold but begin to oscillate as an acoustic wave. Therefore an investigation of the time evolution of the Jeans scale is crucial in particular for the study what is the first collapsed object in the universe.

The Jeans scale is ruled by the sound speed c_s and the free fall time of the object or the expansion time of the universe. After the recombination of hydrogen atoms, the radiation pressure becomes ineffective and equations of state are described by the adiabatic index γ of baryonic matter. If the energy transfer between the baryon and the photon is efficient, we expect isothermal process, i.e., $\gamma = 1$. However, if the time scale of the energy transfer becomes longer than the oscillation time scale of the sound wave, the adiabatic $\gamma = 5/3$ must be achieved. These two γ 's give about factor two difference for the Jeans scales in mass. This point is crucial when we investigate the evolution of the baryon density perturbations smaller than the Jeans scale after recombination. Moreover, the temperature of gas is very close to the radiation temperature at the recombination epoch but gradually separates from it. Eventually, it does change the dependence on the red-shift z from $1+z$ to $(1+z)^2$. In order to obtain an accurate Jeans scale, we need to take into account all these details. In this paper, we formulate the energy transfer

between baryons, electrons and photons from the first principle, i.e., the Boltzmann equation. Similar treatment on photons has been done by Hu, Scott & Silk [7] and Dodelson & Jubas [8]. These works focused on the perturbations of photons which are massless particles in order to get anisotropies of the cosmic microwave background radiation. Here we formulate perturbation equations of baryon-electron fluid which is the massive particle system. The energy transfer between this fluid system and photons is coming through the Compton scattering. We study in detail the sound speed and the Jeans scale after the recombination. It is found that damping of the sound wave is efficient during the change of γ .

In §II, we describe the Boltzmann equation in the perturbed expanding universe. In §III, the equations of the baryon-electron system are derived based on the fluid approximation, by integrating the Boltzmann equation for the momentum space. The integrations of the collisional term which describes the Compton interaction between electrons and photons is summarized in §IV. The perturbation equations of the baryon-electron fluid are obtained in §V. In §VI, we investigate the acoustic motion of the baryon-electron fluid after the recombination in the expanding universe as an application of our formalism. §VII is devoted to summary and discussions. In appendix A, we consider the perturbation equation for the rate equation in order to complete the perturbed equations obtained in §V. In appendix B, we put summary of the physical scales for the cosmological baryon perturbations in the universe. The evolution of the matter temperature is summarized in appendix C.

We will work in units where $c = \hbar = k_B = 1$.

II. BOLTZMANN EQUATION

We write the perturbed space-time to the Friedmann-Robertson-Walker space-time in the Newtonian gauge as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(1 + 2\Psi)dt^2 + \left(\frac{a}{a_0}\right)^2 (1 + 2\Phi)\delta_{ij} dx^i dx^j, \quad (2.1)$$

with introducing the perturbed gravitational potential Ψ and the curvature perturbation Φ . a is the scale factor and suffix 0 indicates the present value. Note that we employ a flat orthogonal coordinate system besides the scale factor as indicated by the Kronecker's delta δ_{ij} for a background space. Ψ satisfies the Poisson equation

$$\nabla^2 \Psi = 4\pi G \rho \left(\frac{a}{a_0}\right)^2 \delta, \quad (2.2)$$

where ρ and δ are total background density and density fluctuation, respectively, and $\Phi = -\Psi$ when the anisotropic stress is negligible, e.g., in the matter-dominated era.

We write next the Boltzmann equation for the distribution function $f_{(\alpha)}(t, x^i, q^i)$ of the (α) -particle as

$$\frac{\partial f_{(\alpha)}}{\partial t} + \frac{\partial f_{(\alpha)}}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f_{(\alpha)}}{\partial q^i} \frac{dq^i}{dt} = C[f_{(\alpha)}], \quad (2.3)$$

with the collision term in the right hand side. This Boltzmann equation is written in terms of the momentum q^i measured by an observer in the cosmological rest frame. In order to solve this equation, we must rewrite the terms dx^i/dt and dq^i/dt in terms of x^i and q^i , which is obtained from the equation of motion of the (α) -particle. To obtain this equation, we write the 4-momentum in the locally orthonormal frame as (q^0, q^i) . Here the energy in this frame q^0 is defined as

$$q^0 := \sqrt{\mathbf{q}^2 + m_{(\alpha)}^2}, \quad (2.4)$$

where $\mathbf{q}^2 = \sum_i (q^i)^2$.

The equation of motion is given from the geodesic equation. However, the geodesic equation of the (α) -particle is commonly written in terms of the 4-momentum p^μ in the frame (2.1), which is defined by $p^\mu = dx^\mu/d\lambda$. Here $\lambda(d\lambda = ds/m_{(\alpha)})$ is the affine parameter. We use the fact that the 4-momentum (q^0, q^i) is related to p^μ , as follows,

$$q^i = \sqrt{g_{ii}} \frac{dx^i}{d\lambda} = \frac{a}{a_0} (1 + \Phi) p^i, \quad (2.5)$$

$$q^0 = \sqrt{-g_{00}} \frac{dx^0}{d\lambda} = (1 + \Psi) p^0. \quad (2.6)$$

Equations (2.5) and (2.6) give the following relations, up to the first order of Ψ and Φ ,

$$\frac{dx^i}{dt} = \frac{p^i}{p^0} = \frac{a_0}{a} (1 + \Psi - \Phi) \frac{q^i}{q^0}, \quad (2.7)$$

and

$$\frac{dq^i}{dt} = \frac{\dot{a}}{a} q^i + \frac{a}{a_0} \left(\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x^j} \frac{dx^j}{dt} \right) p^i + \frac{a}{a_0} (1 + \Phi) \frac{dp^i}{dt}, \quad (2.8)$$

where the over-dot denotes t -differentiation.

On the other hand, the geodesic equation in the leading order of the perturbation derives

$$\frac{dp^i}{dt} = - \left(\frac{a_0}{a} \right)^2 \left[2 \frac{a}{a_0} \frac{\dot{a}}{a} (1 - \Phi) q^i + 2 \frac{a}{a_0} \dot{\Phi} q^i + 2 \Phi_{,j} \frac{q^j q^i}{q^0} + \delta^{ij} \left(\Psi_{,j} q^0 - \Phi_{,j} \frac{\mathbf{q}^2}{q^0} \right) \right], \quad (2.9)$$

where $_{,i} = \partial/\partial x^i$. Hereafter we will omit δ^{ij} and will write as e.g., $\Psi_{,i} = \delta^{ij} \Psi_{,j}$. Inserting this to Eq.(2.8),

$$\frac{dq^i}{dt} = - \frac{a_0}{a} \left[\frac{a}{a_0} \frac{\dot{a}}{a} q^i + \frac{a}{a_0} \dot{\Phi} q^i + \Phi_{,j} \frac{q^j q^i}{q^0} + \Psi_{,i} q^0 - \Phi_{,i} \frac{\mathbf{q}^2}{q^0} \right]. \quad (2.10)$$

Now we can write down the left hand side of the Boltzmann equation (2.3) using the equations (2.7) and (2.10). If we employ the conformal time defined by $(a/a_0)d\eta \equiv dt$ instead of the proper time, it becomes as

$$\begin{aligned} \frac{\partial f_{(\alpha)}}{\partial \eta} + \frac{\partial f_{(\alpha)}}{\partial x^i} (1 - \Phi + \Psi) \frac{q^i}{q^0} \\ + \frac{\partial f_{(\alpha)}}{\partial q^i} \left[- \frac{a'}{a} q^i - \Phi' q^i - \Psi_{,i} q^0 - \Phi_{,j} \frac{q^j q^i}{q^0} + \Phi_{,i} \frac{\mathbf{q}^2}{q^0} \right] = \frac{a}{a_0} C[f_{(\alpha)}], \end{aligned} \quad (2.11)$$

where the prime denotes η -differentiation. Until now, we have assumed only the smallness of perturbations to the homogeneous background.

If we introduce a further assumption that the motion of the (α) -particle is non-relativistic, the Boltzmann equation (2.11) reduces to

$$\frac{\partial f_{(\alpha)}}{\partial \eta} + \frac{\partial f_{(\alpha)}}{\partial x^i} \frac{q^i}{m_{(\alpha)}} + \frac{\partial f_{(\alpha)}}{\partial q^i} \left[- \left(\frac{a'}{a} + \Phi' \right) q^i - \Psi_{,i} m_{(\alpha)} \right] = \frac{a}{a_0} C[f_{(\alpha)}], \quad (2.12)$$

where $v^i = q^i/m_{(\alpha)}$ and terms of $O(\mathbf{v}^2 \times \Psi \text{ or } \Phi)$ and $O(\mathbf{v}^3)$ are omitted. This is a familiar non-relativistic Boltzmann equation in the expanding universe, when the term proportional to Φ' is neglected.

III. FLUID APPROXIMATION

We consider a system of non-relativistic particles (baryons and electrons) and photons which interact only with the electrons. Neutral and ionized hydrogen atoms and neutral helium atoms are taken into account as baryonic components. We further take a single fluid approximation for this baryon and electron system since the time scale of the interaction between them is short enough. Under these assumptions, we here derive the fluid equation from the Boltzmann equation (2.12).

Since the particles are non-relativistic, we take the distribution function of the (α) -particle, where $\alpha = e, H$ and He for the electron, the hydrogen and the helium, as

$$f_{(\alpha)} = S n_{(\alpha)}(\mathbf{x}) \left(\frac{2\pi}{m_{(\alpha)} T_b(\mathbf{x})} \right)^{3/2} \exp \left[\frac{-(\mathbf{q}_{(\alpha)} - m_{(\alpha)} \mathbf{v}_b(\mathbf{x}))^2}{2m_{(\alpha)} T_b(\mathbf{x})} \right], \quad (3.1)$$

with a normalization of

$$\int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3} f_{(\alpha)} = S n_{(\alpha)}(\mathbf{x}). \quad (3.2)$$

Here $T_b(\mathbf{x})$ and $\mathbf{v}_b(\mathbf{x})$ are the temperature and the peculiar velocity of this fluid system which the suffix b denotes. Then it follows that

$$\int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3} q_{(\alpha)}^i f_{(\alpha)} = m_{(\alpha)} S n_{(\alpha)} v_b^i, \quad (3.3)$$

$$\int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3} q_{(\alpha)}^i q_{(\alpha)}^j f_{(\alpha)} = m_{(\alpha)}^2 S n_{(\alpha)} v_b^i v_b^j + m_{(\alpha)} S n_{(\alpha)} T_b \delta^{ij}, \quad (3.4)$$

$$\int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3} \mathbf{q}_{(\alpha)}^2 q_{(\alpha)}^i f_{(\alpha)} = m_{(\alpha)}^3 S n_{(\alpha)} \mathbf{v}_b^2 v_b^i + 5 m_{(\alpha)}^2 S n_{(\alpha)} T_b v_b^i. \quad (3.5)$$

Operating the following integration and the summation for the Boltzmann equation (2.12) with the distribution function (3.1),

$$\sum_{(\alpha)} m_{(\alpha)} \int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3}, \quad (3.6)$$

we obtain the continuity equation

$$\frac{\partial \rho_b}{\partial \eta} + 3 \left(\frac{a'}{a} + \Phi' \right) \rho_b + \frac{\partial}{\partial x^i} (\rho_b v_b^i) = 0 \quad (3.7)$$

for

$$\rho_b(\mathbf{x}) \equiv \sum_{(\alpha)} m_{(\alpha)} S n_{(\alpha)}(\mathbf{x}). \quad (3.8)$$

The collision terms should cancel out by the summation.

Operating the following integration and the summation to Eq.(2.3),

$$\sum_{(\alpha)} \int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3} q_{(\alpha)}^i, \quad (3.9)$$

we get the Euler equation

$$\frac{\partial(\rho_b v_b^i)}{\partial \eta} + 4\left(\frac{a'}{a} + \Phi'\right)\rho_b v_b^i + \frac{\partial(\rho_b v_b^i v_b^j)}{\partial x^j} + \frac{\partial P}{\partial x^i} + \rho_b \frac{\partial \Psi}{\partial x^i} = \Delta V_{\text{Compton}}^i, \quad (3.10)$$

where

$$P \equiv \sum_{(\alpha)} S n_{(\alpha)} T_b, \quad (3.11)$$

and

$$\Delta V_{\text{Compton}}^i \equiv \frac{a}{a_0} \int \frac{d^3 \mathbf{q}_{(e)}}{(2\pi)^3} q_{(e)}^i C[f_{(e)}]_{\text{Compton}}. \quad (3.12)$$

The contributions from the collision term between the baryon and the electron should cancel out while the contribution from the Compton interaction between the electron and the photon, $\Delta V_{\text{Compton}}$, remains. The explicit form of $\Delta V_{\text{Compton}}^i$ is given in the next section.

Operating the following integration and the summation to Eq.(2.3),

$$\sum_{(\alpha)} \frac{1}{2m_{(\alpha)}} \int \frac{d^3 \mathbf{q}_{(\alpha)}}{(2\pi)^3} \mathbf{q}_{(\alpha)}^2, \quad (3.13)$$

we get the energy equation

$$\begin{aligned} & \frac{\partial}{\partial \eta} \left(\rho_b \left(\frac{\mathbf{v}_b^2}{2} + h \right) \right) + 5 \left(\frac{a'}{a} + \Phi' \right) \rho_b \left(\frac{\mathbf{v}_b^2}{2} + h \right) + \frac{\partial}{\partial x^i} \left(v_b^i \rho_b \left(\frac{\mathbf{v}_b^2}{2} + h \right) \right) \\ & - \frac{\partial P}{\partial \eta} - 5 \left(\frac{a'}{a} + \Phi' \right) P + \frac{\partial \Psi}{\partial x^i} v_b^i \rho_b = \Delta E_{\text{Compton}}, \end{aligned} \quad (3.14)$$

where

$$h \equiv \frac{5P}{2\rho_b} = \frac{5 \sum_{(\alpha)} S n_{(\alpha)} T_b}{2 \sum_{(\alpha)} m_{(\alpha)} S n_{(\alpha)}}, \quad (3.15)$$

and

$$\Delta E_{\text{Compton}} \equiv \frac{1}{2m_e} \frac{a}{a_0} \int \frac{d^3 \mathbf{q}_{(e)}}{(2\pi)^3} \mathbf{q}_{(e)}^2 C[f_{(e)}]_{\text{Compton}}. \quad (3.16)$$

IV. COLLISION TERM

Here let us evaluate the collision term between the electron and the photon. The explicit form of $C[f_{(e)}]_{\text{Compton}}$ is

$$\begin{aligned} C[f_{(e)}(\mathbf{q})]_{\text{Compton}} = & \frac{1}{2q^0} \int \int \int \frac{2d^3 \mathbf{p}}{(2\pi)^3 2p^0} \frac{2d^3 \mathbf{p}'}{(2\pi)^3 2p'^0} \frac{2d^3 \mathbf{q}'}{(2\pi)^3 2q'^0} (2\pi)^4 \delta^{(4)}(p + q - p' - q') |M|^2 \\ & \times \left\{ (1 + f_{(\gamma)}(\mathbf{p})) f_{(\gamma)}(\mathbf{p}') f_{(e)}(\mathbf{q}') - (1 + f_{(\gamma)}(\mathbf{p}')) f_{(\gamma)}(\mathbf{p}) f_{(e)}(\mathbf{q}) \right\}, \end{aligned} \quad (4.1)$$

where $f_{(\gamma)}(\mathbf{p})$ is the photon distribution function, \mathbf{p} and \mathbf{p}' are the photon momenta, $p^0 \equiv |\mathbf{p}|$, $p'^0 \equiv |\mathbf{p}'|$ and $\delta^{(4)}(p + q - p' - q') = \delta(p^0 + q^0 - p'^0 - q'^0) \delta^{(3)}(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}')$ is the 4-dimensional Dirac's delta

function. $|M|^2$ is the matrix element summing and averaging over the electron spin and the photon polarization [9] which is described as,

$$|M|^2 = \frac{(4\pi)^2}{2} \alpha_{\text{EM}}^2 \left(\frac{\tilde{p}'}{\tilde{p}} + \frac{\tilde{p}}{\tilde{p}'} - \sin^2 \tilde{\beta} \right). \quad (4.2)$$

Here \tilde{p} and $\tilde{\beta}$ are the photon energy and the photon scattering angle, respectively, in the electron rest frame, and $\alpha_{\text{EM}}^2 = 3m_e^2 \sigma_T / 8\pi$.

Since the explicit integration to get the collision term is very tedious, instead of performing the integration of (4.1), we here consider the Boltzmann equation for the photon distribution function

$$\frac{\partial f_{(\gamma)}}{\partial t} + \frac{\partial f_{(\gamma)}}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f_{(\gamma)}}{\partial p^i} \frac{dp^i}{dt} = C[f_{(\gamma)}]_{\text{Compton}}, \quad (4.3)$$

where

$$\begin{aligned} C[f_{(\gamma)}(\mathbf{p})]_{\text{Compton}} = & \frac{1}{2p^0} \int \int \int \frac{2d^3\mathbf{q}}{(2\pi)^3 2q^0} \frac{2d^3\mathbf{q}'}{(2\pi)^3 2q'^0} \frac{2d^3\mathbf{p}'}{(2\pi)^3 2p'^0} (2\pi)^4 \delta^{(4)}(p + q - p' - q') |M|^2 \\ & \times \frac{1}{2} \left\{ (1 + f_{(\gamma)}(\mathbf{p})) f_{(\gamma)}(\mathbf{p}') f_{(e)}(\mathbf{q}') - (1 + f_{(\gamma)}(\mathbf{p}')) f_{(\gamma)}(\mathbf{p}) f_{(e)}(\mathbf{q}) \right\}. \end{aligned} \quad (4.4)$$

Note the factor of 1/2 in the last line. We should put this factor because we have defined the electron distribution function by Eq.(3.1) or Eq.(4.5), where the spin degree of freedom is summed.

The integration of the collision term Eq.(4.4) has been performed by Dodelson & Jubas [8] and Hu, Scott & Silk [7]. The calculation is based on the assumption that the electron motion is non-relativistic, and is performed by expanding in terms of $O(1/m_e)$. In particular, in the paper by Dodelson & Jubas [8], the integration of the collision term is carried by assuming the same form of the electron distribution function as Eq.(3.1), i.e.,

$$f_{(e)} = n_e(\mathbf{x}) \left(\frac{2\pi}{m_e T_b(\mathbf{x})} \right)^{3/2} \exp \left[\frac{-(\mathbf{q}_e - m_e \mathbf{v}_b(\mathbf{x}))^2}{2m_e T_b(\mathbf{x})} \right], \quad (4.5)$$

and by expressing the photon distribution function in the power expansion of $O(1/m_e)$ up to the second order

$$f_{(\gamma)} = f_{(\gamma)}^{(0)}(p^0) + f_{(\gamma)}^{(1)}(\mathbf{p}) + f_{(\gamma)}^{(2)}(\mathbf{p}). \quad (4.6)$$

Following their result [8] *, the collision term is written in the second order of perturbation as

$$C[f_{(\gamma)}(\mathbf{p})]_{\text{Compton}} = C^{(1)} + C^{(2)}, \quad (4.7)$$

with

$$C^{(1)} = n_e \sigma_T \left[f_{(\gamma)_0}^{(1)} + \frac{1}{2} f_{(\gamma)_2}^{(1)} P_2(\mu) - f_{(\gamma)}^{(1)} - p^0 \frac{\partial f_{(\gamma)}^{(0)}}{\partial p^0} \mu v_b \right], \quad (4.8)$$

and

*Eq.(5.1) in their paper seems to contain a typographical error.

$$\begin{aligned}
C^{(2)} = n_e \sigma_T & \left[f_{(\gamma)_0}^{(2)} + \frac{1}{2} f_{(\gamma)_2}^{(2)} P_2(\mu) - f_{(\gamma)}^{(2)} \right. \\
& + v_b^2 p^0 \frac{\partial f_{(\gamma)}^{(0)}}{\partial p^0} (\mu^2 + 1) + v_b^2 (p^0)^2 \frac{\partial^2 f_{(\gamma)}^{(0)}}{(\partial p^0)^2} \left(\frac{11}{20} \mu^2 + \frac{3}{20} \right) \\
& \left. + \frac{1}{m_e (p^0)^2} \frac{\partial}{\partial p^0} \left\{ (p^0)^4 \left(T_b \frac{\partial f_{(\gamma)}^{(0)}}{\partial p^0} + f_{(\gamma)}^{(0)} (1 + f_{(\gamma)}^{(0)}) \right) \right\} \right], \tag{4.9}
\end{aligned}$$

where

$$f_{(\gamma)_l}^{(i)} = \int_{-1}^1 \frac{d\mu}{2} P_l(\mu) f_{(\gamma)}^{(i)}, \tag{4.10}$$

$$\mu = \frac{\mathbf{v}_b \cdot \mathbf{p}}{|\mathbf{v}_b| |\mathbf{p}|}, \tag{4.11}$$

with $P_l(\mu)$ being a Legendre polynomial.

The energy transfer rate is described as

$$\Delta E_{\text{Compton}} = -\frac{a}{a_0} 2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} p^0 C[f_{(\gamma)}(\mathbf{p})]_{\text{Compton}}. \tag{4.12}$$

Therefore we obtain

$$\Delta E_{\text{Compton}} = -\frac{a}{a_0} \frac{4n_e \sigma_T \rho_{\gamma}^{(0)}}{m_e} \left(\frac{m_e}{3} \mathbf{v}_b^2(\mathbf{x}) - T_{\gamma}^{(0)} + T_b(\mathbf{x}) \right), \tag{4.13}$$

where $T_{\gamma}^{(0)}$ is the background photon temperature and $\rho_{\gamma}^{(0)} = \pi^2 T_{\gamma}^{(0)4}/15$ is the background photon energy density.

The momentum transfer rate can be found from

$$\Delta V_{\text{Compton}} = -\frac{a}{a_0} 2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \mathbf{p} C[f_{(\gamma)}(\mathbf{p})]_{\text{Compton}}, \tag{4.14}$$

and the result of the integration in the leading order of perturbations gives the well known form [5],

$$\Delta V_{\text{Compton}} = \frac{4}{3} \frac{a}{a_0} n_e \sigma_T \rho_{\gamma}^{(0)} (\mathbf{v}_{\gamma} - \mathbf{v}_b). \tag{4.15}$$

V. PERTURBATION EQUATIONS

Electrons(e), neutral and ionized hydrogen atoms(H) and helium atoms(He) are the particle species of the baryon-electron fluid. The number densities for each species are written

$$n_e = x \left(1 - \frac{y_p}{2} \right) n_B, \tag{5.1}$$

$$n_H = \left(1 - y_p \right) n_B, \tag{5.2}$$

$$n_{\text{He}} = \frac{y_p}{4} n_B, \tag{5.3}$$

where y_p is the primordial helium mass fraction, n_H and n_{He} are the number density of neutral and ionized hydrogen and helium, respectively, $n_B \equiv n_H + 4n_{\text{He}}$ is the total baryon number density and x is an ionization fraction defined as $x \equiv n_e / (n_H + 2n_{\text{He}})$. Then,

$$\rho_b = \sum_{(\alpha)} m_{(\alpha)} S n_{(\alpha)} \simeq m_p n_B \quad (5.4)$$

$$P = \sum_{(\alpha)} S n_{(\alpha)} T_b = \left(1 + x - \frac{y_p}{2} \left(x + \frac{3}{2}\right)\right) n_B T_b, \quad (5.5)$$

where m_p is the proton mass.

First let us summarize the equations for the baryon-electron fluid in the expanding universe. The continuity equation is (3.7). The Euler equation (3.10) reduces to

$$\frac{\partial v_b^i}{\partial \eta} + \left(\frac{a'}{a} + \Phi'\right) v_b^i + v_b^j \frac{\partial v_b^i}{\partial x^j} + \frac{1}{\rho_b} \frac{\partial P}{\partial x^i} + \frac{\partial \Psi}{\partial x^i} = \frac{a}{a_0} \frac{n_e \sigma_T}{R} (v_\gamma^i - v_b^i), \quad (5.6)$$

with the use of Eq.(3.7) and $R \equiv 3\rho_b/4\rho_\gamma^{(0)}$. The energy equation (3.14) reduces to

$$\begin{aligned} \frac{\partial}{\partial \eta} \left(\frac{\mathbf{v}_b^2}{2} + h \right) + 2 \left(\frac{a'}{a} + \Phi' \right) \left(\frac{\mathbf{v}_b^2}{2} + h \right) + v_b^i \frac{\partial}{\partial x^i} \left(\frac{\mathbf{v}_b^2}{2} + h \right) - \frac{1}{\rho_b} \frac{\partial P}{\partial \eta} - \frac{5}{\rho_b} \left(\frac{a'}{a} + \Phi' \right) P \\ + \frac{\partial \Psi}{\partial x^i} v_b^i = 4 \frac{a}{a_0} \frac{x \left(1 - y_p/2\right) \sigma_T \rho_\gamma^{(0)}}{m_e m_p} \left(T_\gamma^{(0)} - T_b(\mathbf{x}) - \frac{m_e}{3} \mathbf{v}_b^2 \right), \end{aligned} \quad (5.7)$$

by using Eqs.(3.7), (5.1) and (5.4). We supplement the equation of state,

$$h = \frac{5P}{2\rho_b}, \quad P = \frac{\rho_b}{m_p} \left(1 + x - \frac{y_p}{2} \left(x + \frac{3}{2}\right)\right) T_b. \quad (5.8)$$

Now we solve the above equations by a perturbative method assuming a small deviations from the uniformity. Define the perturbative expansions as follows,

$$\rho_b = \rho_b^{(0)}(\eta)(1 + \delta_b(\eta, \mathbf{x})), \quad (5.9)$$

$$h = h^{(0)}(\eta)(1 + \Delta_h(\eta, \mathbf{x})), \quad (5.10)$$

$$P = P^{(0)}(\eta)(1 + \Delta_P(\eta, \mathbf{x})), \quad (5.11)$$

$$T_b = T_b^{(0)}(\eta)(1 + \Delta_{T_b}(\eta, \mathbf{x})), \quad (5.12)$$

$$x = x^{(0)}(\eta) + \delta x(\eta, \mathbf{x}), \quad (5.13)$$

together with $\mathbf{v}_b = \mathbf{v}_b(\eta, \mathbf{x})$.

The zero-th order equations of (3.7) and (5.7) are

$$\rho_b^{(0)'} + 3 \frac{a'}{a} \rho_b^{(0)} = 0, \quad (5.14)$$

and

$$\frac{\partial h^{(0)}}{\partial \eta} + 2 \frac{a'}{a} h^{(0)} - \frac{1}{\rho_b^{(0)}} \left(\frac{\partial P^{(0)}}{\partial \eta} + 5 \frac{a'}{a} P^{(0)} \right) = 4 \frac{a}{a_0} \frac{x^{(0)} \left(1 - y_p/2\right) \sigma_T \rho_\gamma^{(0)}}{m_e m_p} \left(T_\gamma^{(0)} - T_b^{(0)} \right), \quad (5.15)$$

respectively. Eq.(5.15) reduces to [10]

$$T_b^{(0)'} + 2 \frac{a'}{a} T_b^{(0)} = \eta_E^{-1} (T_\gamma^{(0)} - T_b^{(0)}), \quad (5.16)$$

where we have neglected the term proportional to $x^{(0)'}T_b^{(0)\dagger}$ and used the zero-th order equation of state,

$$h^{(0)} = \frac{5P^{(0)}}{2\rho_b^{(0)}}, \quad P^{(0)} = \frac{\rho_b^{(0)}}{m_p} \left(1 + x^{(0)} - \frac{y_p}{2} \left(x^{(0)} + \frac{3}{2} \right) \right) T_b^{(0)}. \quad (5.17)$$

We defined the Compton energy transfer time scale η_E as

$$\eta_E^{-1} = \frac{8}{3} \frac{a}{a_0} \frac{x^{(0)} \left(1 - y_p/2 \right) \sigma_T \rho_\gamma^{(0)}}{m_e \left(1 + x^{(0)} - (x^{(0)} + 3/2)y_p/2 \right)}. \quad (5.18)$$

The solution of Eq.(5.16) is discussed in appendix C.

The perturbed equations of (3.7) and (5.6) are

$$\delta_b' + 3\Phi' + \frac{\partial v_b^i}{\partial x^i} = 0, \quad (5.19)$$

and

$$v_b^{i'} + \frac{a'}{a} v_b^i + \frac{P^{(0)}}{\rho_b^{(0)}} \frac{\partial \Delta_P}{\partial x^i} + \frac{\partial \Psi}{\partial x^i} = \frac{a}{a_0} \frac{n_e \sigma_T}{R} \left(v_\gamma^i - v_b^i \right). \quad (5.20)$$

The perturbed equation of (5.7) yields

$$\begin{aligned} \Delta_h' + \left(\frac{5}{3} \frac{h^{(0)'}}{h^{(0)}} - \frac{2}{3} \frac{P^{(0)'}}{P^{(0)}} \right) \Delta_h - \frac{2}{3} \delta_b' \\ = \eta_E^{-1} \left(-\Delta_h + \frac{\delta x}{x^{(0)}} \frac{T_\gamma^{(0)} - T_b^{(0)}}{T_b^{(0)}} + \frac{\left(1 - y_p/2 \right) \delta x}{1 + x^{(0)} - (x^{(0)} + 3/2)y_p/2} \right), \end{aligned} \quad (5.21)$$

where we have used the perturbative part of the equation of state as follows,

$$\Delta_h = \Delta_P - \delta_b, \quad (5.22)$$

and

$$\Delta_P = \delta_b + \Delta_{T_b} + \frac{\left(1 - y_p/2 \right) \delta x}{1 + x^{(0)} - (x^{(0)} + 3/2)y_p/2}. \quad (5.23)$$

In order to complete the perturbation equations, we need an equation to specify the evolution of the perturbation of the ionization rate δx in equations (5.21) and (5.23), which we have considered in the appendix A. In the early phase of the recombination the terms proportional to δx become important.

[†]As we will see in the below, the matter temperature follows the photon temperature in the recombination regime. This is because the energy transfer time scale η_E is small enough. After recombination, the fraction of the residual ionization $x^{(0)}$ is almost fixed, and the time variation of $x^{(0)}$ is small. This is the reason why we neglected the term proportional to $x^{(0)'}T_b^{(0)}$.

VI. SOUND WAVES AFTER THE RECOMBINATION

As an application of the basic relations obtained in the previous sections, we consider the behavior of the sound wave of the baryon-electron fluid after the recombination epoch. Neglecting a perturbation of the gravitational potential, we get basic equations from Eqs.(5.19), (5.20), (5.21) as

$$\delta_b' + \frac{\partial v_b^i}{\partial x^i} = 0, \quad (6.1)$$

$$v_b^{i'} + \frac{a'}{a} v_b^i + \frac{P^{(0)}}{\rho_b^{(0)}} \frac{\partial \Delta_P}{\partial x^i} = 0, \quad (6.2)$$

and

$$\Delta_h' + \left(\frac{5}{3} \frac{h^{(0)'}}{h^{(0)}} - \frac{2}{3} \frac{P^{(0)'}}{P^{(0)}} \right) \Delta_h - \frac{2}{3} \delta_b' = -\eta_E^{-1} \Delta_h. \quad (6.3)$$

Here we have also neglected a perturbation of the ionization fraction and an interaction with the photon in the Euler equation because the momentum transfer rate is so small after the decoupling. The wave equation is derived from Eqs.(6.1) and (6.2), together with Eq.(5.22), as

$$\delta_b'' + \frac{a'}{a} \delta_b' - \frac{P^{(0)}}{\rho_b^{(0)}} \frac{\partial^2}{\partial \mathbf{x}^2} (\Delta_h + \delta_b) = 0. \quad (6.4)$$

This must be coupled with the energy equation (6.3) in order to investigate the sound oscillation of the baryon-electron fluid.

Now let us omit the time-dependence of the background quantities. When the time scale of the sound oscillation is shorter than the Hubble time, this is a good approximation. Then Eqs.(6.4) and (6.3) reduce to simple equations

$$\delta_b(k, \eta)'' + \frac{P^{(0)}}{\rho_b^{(0)}} k^2 (\Delta_h(k, \eta) + \delta_b(k, \eta)) = 0, \quad (6.5)$$

$$\Delta_h(k, \eta)' = \frac{2}{3} \delta_b(k, \eta)' - \eta_E^{-1} \Delta_h(k, \eta), \quad (6.6)$$

where we took the Fourier mode expansion by setting $\delta_b = \delta_b(k, \eta) e^{i\mathbf{k} \cdot \mathbf{x}}$, and $\Delta_h = \Delta_h(k, \eta) e^{i\mathbf{k} \cdot \mathbf{x}}$. The above equations yield

$$\eta_E \frac{d}{d\eta} \left(\frac{d^2 \delta_b(k, \eta)}{d\eta^2} + c_f^2 k^2 \delta_b(k, \eta) \right) + \left(\frac{d^2 \delta_b(k, \eta)}{d\eta^2} + c_e^2 k^2 \delta_b(k, \eta) \right) = 0. \quad (6.7)$$

Here we defined

$$c_f^2 = \frac{5}{3} \frac{P^{(0)}}{\rho_b^{(0)}}, \quad c_e^2 = \frac{P^{(0)}}{\rho_b^{(0)}}. \quad (6.8)$$

As is well known, c_f is the sound speed for the adiabatic state and c_e is the one for the isothermal state.

Taking the wave solution $\delta_b(k, \eta) \propto e^{-i\omega\eta}$, we get the following dispersion relation,

$$-i\eta_E \omega (\omega^2 - c_f^2 k^2) + \omega^2 - c_e^2 k^2 = 0. \quad (6.9)$$

In order to solve this equation, it is convenient to introduce the variable ϖ such as $\omega = i\varpi$ and we have

$$c_e k \eta_E \left(\frac{\varpi}{c_e k} \right) \left(\left(\frac{\varpi}{c_e k} \right)^2 + \frac{5}{3} \right) + \left(\frac{\varpi}{c_e k} \right)^2 + 1 = 0. \quad (6.10)$$

The sound speed c_s and the adiabatic index γ are defined as

$$c_s = -\text{Im}\left[\frac{\varpi}{k}\right], \quad \gamma = \frac{c_s^2}{c_e^2}. \quad (6.11)$$

Therefore we need to solve the cubic equation (6.10) in order to get the sound speed of the baryon-electron system.

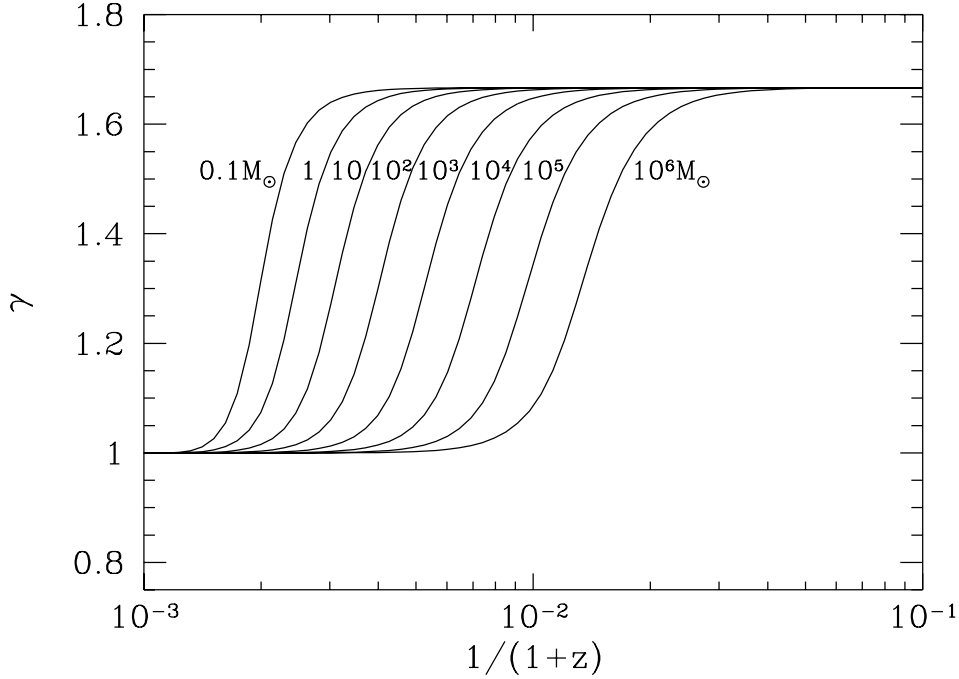


FIG. 1. The behaviors of γ as a function of $1/(1+z)$. Each lines for the mass scales $M = 1M_\odot, 10^3M_\odot, 10^6M_\odot, 10^9M_\odot, 10^{12}M_\odot$, respectively, with the cosmological parameters, $h = 0.5$, $\Omega_0 = 1.0$, $\Omega_B = 0.1$. γ is changed in the earlier stage from 1 to $5/3$ for the smaller scale perturbations.

A cubic equation can be solved exactly in an analytic form while the expression is complicated. It may be instructive to show the solution of Eq.(6.10) in a simple form with some approximation. Expanding the solution around the adiabatic state, i.e., expanding in terms of ϵ by setting the solution $\omega/k = c_f(1 + \epsilon)$, we find

$$\frac{\omega}{k} \simeq c_f \left(1 - \frac{1 + ikc_f\eta_E}{5(1 + (kc_f\eta_E)^2)} \right). \quad (6.12)$$

It is interesting to clarify the behavior of the adiabatic index γ in the expanding universe after the recombination, which we demonstrate as an example of the usefulness of our formalism in the below. As is clear from Eq.(6.9), the adiabatic index γ is ruled by the ratio of the Compton energy transfer time scale η_E to the sound oscillation time scale. If $\eta_E\omega \ll 1$, $\omega \simeq c_e k$ and $\gamma \simeq 1$. On the other hand, if $\eta_E\omega \gg 1$, $\omega \simeq c_f k$ and $\gamma \simeq 5/3$. Therefore the sound speed depends on scales of the perturbations, and it is expected that γ is changed from 1 to $5/3$ as the universe expands.

Fig.1 shows γ as a function of $1/(1+z)$ by directly solving Eq.(6.10). The ionization fraction is calculated properly by solving the recombination process in the expanding universe with the cosmological

parameters in the figure caption [11,12]. As is expected, γ is changed from 1 to 5/3 in the earlier stage of the universe for smaller scale perturbations.

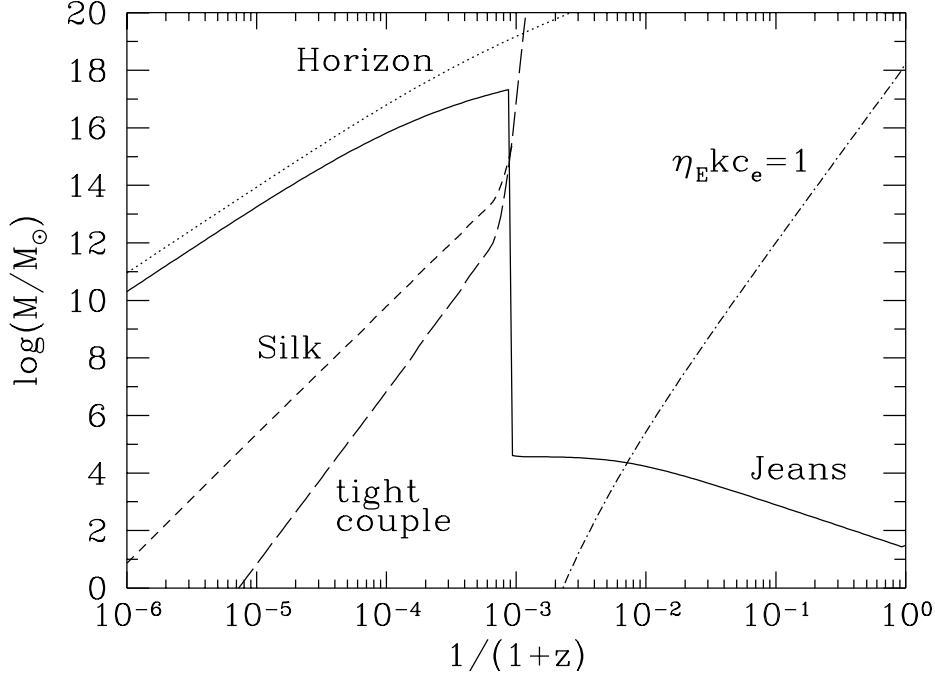


FIG. 2. The physical mass scales for the baryon perturbations. The cosmological parameters taken here are same as those in Fig.1. The definitions of these scales are summarized in appendix B. In the figure, ‘Horizon’ means the horizon scale, ‘Silk’ does the diffusion damping scale, ‘tight couple’ does the breaking scale of the tight coupling approximation of baryon and photon fluids, ‘Jeans’ does the Jeans scale.

To get a physical insight, we refer to the familiar illustrative figure, Fig.2, which gives temporal variations of various physical sizes in the expanding universe. The definitions of the curves are summarized in appendix B. (See also ref. [13].) The adiabatic index γ after the recombination is changed when the Compton energy transfer time is equal to the sound oscillation time scale. In this figure, we show the line on which the two time scales are equal, i.e., $\eta_E k c_e = 1$. Note that we use the (baryon) mass scale in the unit of the solar mass instead of k , with employing the relation $M = (4\pi\rho_b/3)(\pi a/k a_0)^3$. The mass and the comoving wave number which satisfy the relation $\eta_E k c_e = 1$ are expressed (see also appendix B),

$$k = \frac{9.2x^{(0)}(1-y_p/2)}{(\mu^3 T_b^{(0)})^{1/2}} (1+z)^3 \text{ Mpc}^{-1}, \quad (6.13)$$

$$M = \frac{4.6 \times 10^{10} (\mu^3 T_b^{(0)})^{3/2}}{x^{(0)3} (1-y_p/2)^3} (1+z)^{-9} \Omega_b h^2 M_\odot, \quad (6.14)$$

where we set the temperature of the microwave background at present $T_\gamma^{(0)}(t_0) = 2.726\text{K}$, T_b is the baryon temperature in unit of Kelvin, and $\mu = 1 - 3y_p/4$.

As is shown in Fig.2, the Jeans scale after the recombination has the plateau. In this stage the energy transfer between background photons and the baryon fluid is effective through the residual electrons, and the matter temperature follows the photon temperature. As the universe expands, however, the energy

transfer time rises above the Hubble expansion time. After that epoch the matter temperature cools adiabatically and drops as $T_b \propto 1/a^2$. The broken corner of the plateau is the critical time that the two time scales become equal, where this epoch is roughly estimated as $z \simeq 1000(\Omega_b h^2)^{2/5}$ [10] (see also appendix C).

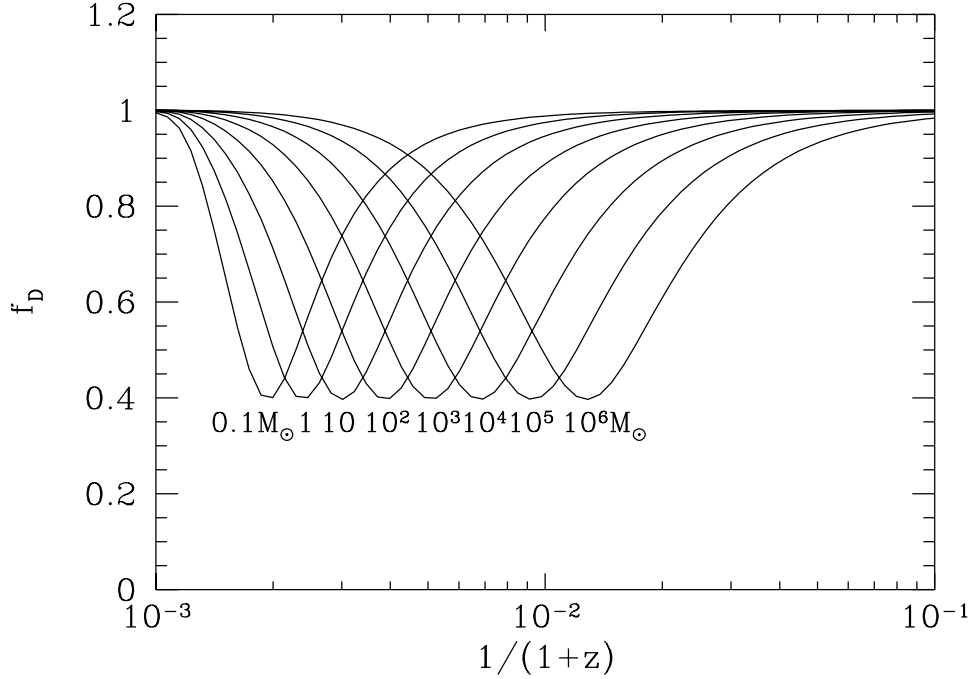


FIG. 3. The behaviors of f_D as the function of $1/(1+z)$. Each lines are for the mass scales, $M = 1M_\odot, 10^3M_\odot, 10^6M_\odot, 10^9M_\odot, 10^{12}M_\odot$, respectively. The cosmological parameters are same as those in Fig.1 The damping phenomenon occurs when the γ is changed from 1 to $5/3$.

The line of $\eta_E k c_e = 1$ crosses at the broken corner of the plateau of the Jeans mass scale. This necessarily happens because of the following reason. The Jeans scale is the scale at which the sound oscillation time is equal to the free fall time of the perturbation or the Hubble expansion time. Since the broken corner of the plateau is the epoch when the Hubble expansion time is equal to the energy transfer time through the Compton interaction between the background photons and the baryon-electron fluid, the cross point of two lines is the point when the sound oscillation time, the Hubble expansion time and the energy transfer time become all same.

Now let us discuss an interesting result derived from the dispersion relation (6.9). As it is apparent from the approximated solution, eq.(6.12), ω generally has an imaginary part. This imaginary part of ω represents the exponential damping of the wave oscillation if $\text{Im } \omega < 0$. Since the period of the sound oscillation is $T = 2\pi/|\text{Re } \omega|$, the damping factor for the amplitude during one period is

$$f_D = \exp \left[-2\pi \left| \frac{\text{Im } \omega}{\text{Re } \omega} \right| \right]. \quad (6.15)$$

We show the behavior of f_D in Fig.3, in the similar way to Fig.1 as the function of $1/(1+z)$. As is shown in Fig.3, this damping phenomenon may have an effect on the evolution of baryon perturbations. We should be notice that our discussions here are based on the assumption that the solutions have the wave form. In other word, this damping process is effective only inside the Jeans scale.

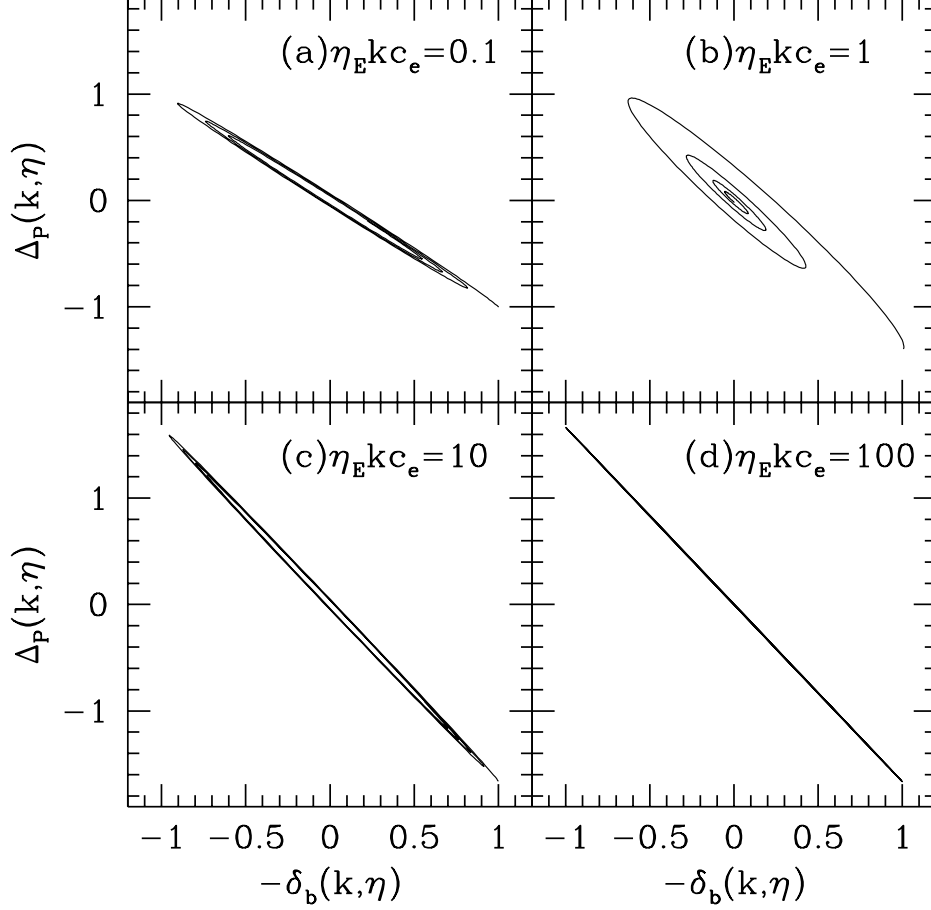


FIG. 4. The trajectory of the solution on $(\Delta_P - \delta_b)$ -plane. The horizontal axis is the baryon density perturbation $-\delta_b(k, \eta)$, and the vertical axis is the pressure perturbation $\Delta_P(k, \eta)$. We take $\eta_E k c_e = 0.1, 1, 10$ and 100 .

In order to understand this damping mechanism, we show another aspect of the solutions of Eq.(6.7). Introducing $\omega_R = \text{Re}\omega$, $\omega_I = \text{Im}\omega$, we take the solution

$$\delta_b(k, \eta) = -\cos(\omega_R \eta) e^{-\omega_I \eta}. \quad (6.16)$$

The initial value is $\delta_b(k, \eta = 0) = -1$. Here ω is obtained by solving the dispersion relation Eq.(6.9). For $\eta_E k c_e \gg 1$ and $\eta_E k c_e \ll 1$, solutions are mere harmonic oscillations with no damping. Therefore the solutions around $\eta_E k c_e \simeq 1$ should be investigated. By employing Eqs. (6.5) and (5.22), the trajectories of the solutions on $(\Delta_P - \delta_b)$ -plane are shown in Fig.4. The horizontal axis is $-\delta_b(k, \eta)$ and the vertical axis is $\Delta_P(k, \eta)$. The non-dimensional quantity $\eta_E k c_e$ is the unique physical parameter of the equation. We have chosen $\eta_E k c_e = 0.1, 1, 10$, and 100 in Fig.4.

From the relation $-\delta_b = \delta V/V$, where V is the volume for unit particle number, the horizontal axis can be regarded as the change of volume per unit particle number. Thus we can regard the trajectories in Fig.4 in the similar way to the thermodynamical cycles in (pressure-volume)-plane. The damping of the oscillation is in proportion to the deviation in one cycle. The most significant damping occurs when $\eta_E k c_e = 1$ (Fig.4(b)). The damping does not happen if the deviation in one cycle is negligible as we can see in Fig.4(d).

VII. SUMMARY AND DISCUSSIONS

We have formulated equations for a fluid system with the electron, neutral and ionized hydrogen atoms and neutral helium atoms with taking into account the energy transfer between the background photons and the residual electrons through the Compton interaction. Using this formulation, we have studied the time evolution of the sound speed and the Jeans scale after the recombination. We found that the behavior of the adiabatic index γ after the recombination is controlled by the ratio of the Compton energy transfer time scale to the sound oscillation time scale. Then γ (or the sound speed) depends on scales of the perturbations, and is changed from 1 to 5/3 in earlier stage for smaller scale perturbations. This formalism enables us to calculate the linear evolution of the very small scale baryon density perturbations [13].

We have also discussed the small damping feature of the sound oscillation when γ changes from 1 to 5/3. This effect works inside the Jeans scale and seems to be negligible on scales $M \gtrsim 10^6 M_\odot$. If early reionization occurs, however, the ionization fraction and the Jeans scale increase. And γ changes from 5/3 to 1. The damping of the baryon perturbation would be effective on larger scales during the process of the reionization of the universe. It has been pointed out that the neutral gas cloud could have an instability in a specific photon background [14]. Since we have not consider the specific situation in our paper, an unstable mode does not appear in our equations. In the present paper we have also neglected perturbations to the ionization fraction. Future work may be required on this point.

It will be important to quantify the limitation of our formalism. Our basic assumption is the fact that the baryon-electron system is treated as tightly coupled single-fluid. This assumption becomes not being correct when the collision time scale of neutral interaction rises above the Hubble expansion time. This epoch can be estimated in the following way [15]. The matter temperature can be written as $T_b = 4.5 \times 10^{-3} (\Omega_b h^2)^{-2/5} (1+z)^2$ [K], after the energy transfer through the Compton interaction becomes ineffective (see appendix C). Then the mean free time for collision between neutral hydrogen atoms is $t_c \simeq 1/(n_B \sigma v)$, where σ is the cross section of neutral atoms $\sigma \simeq \pi r_B^2$ (r_B is the Bohr radius), and $v = \sqrt{3T_b/m_H}$. The ratio of t_c to the Hubble expansion time is therefore expressed as $t_c H \simeq 3.1 (\Omega_0 h^2)^{1/2} (\Omega_b h^2)^{-4/5} (1+z)^{-5/2}$. Eventually we get the red-shift at $t_c H = 1$ as $(1+z) \simeq 1.6 (\Omega_0 h^2)^{0.2} (\Omega_b h^2)^{-0.32}$. This is small enough as long as we consider the linear stage of the density perturbations.

Numerical simulations and semi-analytic calculations of structure formation employ the linear matter power spectrum as their initial conditions. This linear matter spectrum is usually calculated without taking into account the baryon pressure term after the recombination. It is appropriate, however, only if the structures which are larger than the Jeans scale are considered. According to the hierarchical clustering scenario, smaller objects are formed earlier than larger ones. We expect the scale of the first collapsing object is very close to the Jeans scale. Therefore very accurate estimate of the matter spectrum including the baryon pressure term is necessary to understand the early formation of bound objects [13]. Once the first collapsing objects are formed, they may cool down through formation of hydrogen molecules and may fragment into smaller radiating objects like stars and/or quasars [3, 16–18]. The first radiating objects formed Strömgren sphere around them [19] and may eventually ionize all the surrounding gas by UV radiation. This reionization process changes the Jeans scale [20]. After that, full numerical or semi-analytic calculations including collapsing objects are required.

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APPENDIX A: RATE EQUATION

In order to complete the fluid equations we should add the rate equation. For simplicity, we neglect the helium fraction, i.e., $n_B = n_H$, and neglect the helium recombination process in the rate equation. Then, the rate equation is [10]

$$\frac{\partial x}{\partial \eta} = -\langle \sigma v \rangle \frac{a}{a_0} n_B C \left(x^2 - (1-x) \frac{x_{\text{eq}}^2}{1-x_{\text{eq}}} \right), \quad (\text{A1})$$

where $\langle \sigma v \rangle$ is the rate coefficient for recombination to excited states, x_{eq} is the equilibrium ionization fraction which is given by the Saha equation

$$\frac{x_{\text{eq}}^2}{1-x_{\text{eq}}} = \frac{m_p}{\rho_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-13.6\text{eV}/T_b}, \quad (\text{A2})$$

and

$$C = \frac{1 + K \Lambda n_{1s}}{1 + K(\Lambda + \beta_e) n_{1s}}. \quad (\text{A3})$$

Here n_{1s} is the number density of hydrogen in the electron ground state, for which we approximate $n_{1s} = n_B(1-x)$, Λ is the decay rate from the excited state, $\beta_e = \langle \sigma v \rangle (m_e T_b / 2\pi)^{3/2} e^{-3.4\text{eV}/T_e}$ and $K = (a/\dot{a}) \lambda_\alpha^3 / 8\pi$ with λ_α being the Lyman alpha photon wave length.

The Zero-th order equations of the rate equation and the Saha equation are the same forms of Eqs.(A1) and (A2) replaced the variables with the zero-th order quantities.

Next we consider the perturbation of the rate equation. We define

$$x_{\text{eq}} = x_{\text{eq}}^{(0)}(\eta) + \delta x_{\text{eq}}(\eta, \mathbf{x}), \quad (\text{A4})$$

then

$$\begin{aligned} \frac{d\delta x}{d\eta} = \frac{a}{a_0} \langle \sigma v \rangle n_{\text{B}}^{(0)} C \left[\left(x^{(0)2} - \frac{(1-x^{(0)})x_{\text{eq}}^{(0)2}}{1-x_{\text{eq}}^{(0)}} \right) \left(\frac{\delta C}{C} + \delta_{\text{b}} \right) \right. \\ \left. + \left(2x^{(0)} + \frac{x_{\text{eq}}^{(0)2}}{1-x_{\text{eq}}^{(0)}} \right) \delta x - \frac{(1-x^{(0)})x_{\text{eq}}^{(0)}(2-x_{\text{eq}}^{(0)})}{(1-x_{\text{eq}}^{(0)})^2} \delta x_{\text{eq}} \right], \end{aligned} \quad (\text{A5})$$

where

$$\frac{\delta C}{C} = \frac{-K\beta_{\text{e}}n_{\text{B}}^{(0)}\left((1-x^{(0)})\delta_{\text{b}} - \delta x\right)}{\left(1 + K\Lambda n_{\text{B}}^{(0)}(1-x^{(0)})\right)\left(1 + K(\Lambda + \beta_{\text{e}})n_{\text{B}}^{(0)}(1-x^{(0)})\right)}. \quad (\text{A6})$$

We have assumed that $\langle \sigma v \rangle$ and β_{e} are constant.

The perturbation of the Saha equation can be written as

$$\delta x_{\text{eq}} = \left. \frac{\partial x_{\text{eq}}(P^{(0)}, \rho_{\text{b}}^{(0)})}{\partial \rho_{\text{b}}^{(0)}} \right|_{P^{(0)}} \rho_{\text{b}}^{(0)} \delta_{\text{b}} + \left. \frac{\partial x_{\text{eq}}(P^{(0)}, \rho_{\text{b}}^{(0)})}{\partial P^{(0)}} \right|_{\rho_{\text{b}}^{(0)}} P^{(0)} \Delta_P. \quad (\text{A7})$$

APPENDIX B: PHYSICAL SCALES

In this appendix, we summarize the physical scales which are important for the evolution of the baryon perturbations on small scales. The definitions of the physical scales in Fig.2 are given. Mass scales are defined by the amount of baryonic components inside the systems. Here we have set that the temperature of the microwave background at present $T_{\gamma}(t_0) = 2.726\text{K}$. We write f_{ν} as the neutrino fraction of the energy density in the massless particles, and $f_{\nu} = 0.405$ in case of the standard three families of massless neutrinos.

First of all, we use the notation for the physical wave number (length) and the comoving wave number (length) as

$$k^{\text{comv}} = \left(\frac{a}{a_0}\right) k^{\text{phys}}, \quad \lambda^{\text{comv}} = \left(\frac{a_0}{a}\right) \lambda^{\text{phys}} = 2\pi/k^{\text{comv}}. \quad (\text{B1})$$

It will be useful to give the relation between the red-shift z and the scale factor normalized at the matter radiation equality a_{eq} ,

$$\frac{a}{a_{\text{eq}}} = 4.04 \times 10^4 (1 - f_{\nu}) \Omega_0 h^2 (1 + z)^{-1}. \quad (\text{B2})$$

1. Horizon Scale

We define the horizon wave number and the horizon mass as

$$\frac{1}{k_{\text{H}}^{\text{comv}}} = \eta, \quad (\text{B3})$$

$$M_{\text{H}} = \frac{4\pi\rho_{\text{b}}}{3} \left(\frac{\lambda_{\text{H}}^{\text{phys}}}{2}\right)^3 = \frac{4\pi\rho_{\text{b}}}{3} \left(\frac{\pi}{k_{\text{H}}^{\text{phys}}}\right)^3, \quad (\text{B4})$$

which derive

$$k_{\text{H}}^{\text{comv}} = 3.35 \times 10^{-2} \left(\sqrt{1 + \frac{a}{a_{\text{eq}}}} - 1 \right)^{-1} (1 - f_{\nu})^{1/2} \Omega_0 h^2 \text{ Mpc}^{-1}, \quad (\text{B5})$$

$$M_{\text{H}} = 9.57 \times 10^{17} \left(\sqrt{1 + \frac{a}{a_{\text{eq}}}} - 1 \right)^3 (1 - f_{\nu})^{-3/2} \Omega_{\text{b}} h^2 (\Omega_0 h^2)^{-3} M_{\odot}. \quad (\text{B6})$$

2. Jeans scale before the recombination

We define the Jeans wave length (wave number) and the Jeans mass as

$$\lambda_{\text{J}}^{\text{phys}} = \frac{2\pi}{k_{\text{J}}^{\text{phys}}} = \sqrt{\frac{\pi c_s^2}{G(\rho_{\text{m}} + \rho_{\gamma})}}, \quad (\text{B7})$$

$$M_{\text{J}} = \frac{4\pi\rho_{\text{b}}}{3} \left(\frac{\lambda_{\text{J}}^{\text{phys}}}{2} \right)^3 = \frac{4\pi\rho_{\text{b}}}{3} \left(\frac{\pi}{k_{\text{J}}^{\text{phys}}} \right)^3, \quad (\text{B8})$$

where

$$c_s^2 = \frac{1}{3(1 + R)}, \quad (\text{B9})$$

and $\rho_{\text{m}} = \rho_{\text{b}} + \rho_{\text{dm}}$ with ρ_{dm} being the energy density of the dark component. Then we have

$$k_{\text{J}}^{\text{comv}} = 1.42 \times 10^{-1} \left(\frac{a_{\text{eq}}}{a} + (1 - f_{\nu}) \frac{a_{\text{eq}}^2}{a^2} \right)^{1/2} (1 + R)^{1/2} (1 - f_{\nu})^{1/2} \Omega_0 h^2 \text{ Mpc}^{-1}, \quad (\text{B10})$$

$$M_{\text{J}} = 1.25 \times 10^{16} \left(\frac{a_{\text{eq}}}{a} + (1 - f_{\nu}) \frac{a_{\text{eq}}^2}{a^2} \right)^{-3/2} (1 + R)^{-3/2} (1 - f_{\nu})^{-3/2} (\Omega_0 h^2)^{-3} \Omega_{\text{b}} h^2 M_{\odot}. \quad (\text{B11})$$

3. Jeans scale after the recombination

We can define the Jeans scale after the recombination as

$$\lambda_{\text{J}}^{\text{phys}} = \frac{2\pi}{k_{\text{J}}^{\text{phys}}} = \sqrt{\frac{\pi c_s^2}{G\rho_{\text{m}}}}, \quad (\text{B12})$$

with

$$c_s^2 = \gamma \frac{P^{(0)}}{\rho_{\text{b}}^{(0)}} = 9.18 \times 10^{-14} \gamma \mu T_{\text{b}}^{(0)}, \quad (\text{B13})$$

where $\mu = 1 - 3y_{\text{p}}/4$ for $x^{(0)} \ll 1$, and T_{b} is the matter temperature in unit of Kelvin. Then we have

$$k_{\text{J}}^{\text{comv}} = 2.71 \times 10^5 \left(\gamma \mu T_{\text{b}} \frac{a}{a_{\text{eq}}} \right)^{-1/2} (1 - f_{\nu})^{1/2} \Omega_0 h^2 \text{ Mpc}^{-1}, \quad (\text{B14})$$

$$M_{\text{J}} = 1.81 \times 10^{-3} \left(\gamma \mu T_{\text{b}} \frac{a}{a_{\text{eq}}} \right)^{3/2} (1 - f_{\nu})^{-3/2} (\Omega_0 h^2)^{-3} \Omega_{\text{b}} h^2 M_{\odot}. \quad (\text{B15})$$

4. Diffusion damping scale

We define the diffusion damping scales as [12]

$$\left(\frac{1}{k_D^{\text{comv}}}\right)^2 = \frac{1}{6} \int \frac{1}{\dot{\tau}} d\eta \frac{R^2 + 4(1+R)/5}{(1+R)^2}, \quad (\text{B16})$$

$$M_D = \frac{4\pi}{3} \rho_b \left(\frac{\pi}{k_D^{\text{phys}}}\right)^3, \quad (\text{B17})$$

where $\dot{\tau} = n_e \sigma_T (a/a_0)$. For $R \ll 1$, we get

$$k_D^{\text{comv}} = 1.38 \times 10^2 \left(\frac{u(a/a_{\text{eq}})}{x^{(0)}(1-y_p/2)}\right)^{-1/2} (1-f_\nu)^{5/4} (\Omega_0 h^2)^{3/2} (\Omega_b h^2)^{1/2} \text{ Mpc}^{-1}, \quad (\text{B18})$$

$$M_D = 1.38 \times 10^7 \left(\frac{u(a/a_{\text{eq}})}{x^{(0)}(1-y_p/2)}\right)^{3/2} (1-f_\nu)^{-15/4} (\Omega_0 h^2)^{-9/2} (\Omega_b h^2)^{-1/2} M_\odot, \quad (\text{B19})$$

where $u(y) = (\sqrt{1+y}(16-8y+6y^2)-16)/15$.

5. Breaking Scale of the tight coupling approximation

The breaking scale of the tight coupling approximation is defined by $1/k_{\text{BR}}^{\text{phys}} = 1/n_e \sigma_T$, i.e.,

$$k_{\text{BR}}^{\text{comv}} = n_e \sigma_T \frac{a}{a_0}, \quad (\text{B20})$$

$$M_{\text{BR}} = \frac{4\pi}{3} \rho_b \left(\frac{\pi}{k_{\text{BR}}^{\text{phys}}}\right)^3. \quad (\text{B21})$$

Then we have

$$k_{\text{BR}}^{\text{comv}} = 3.77 \times 10^4 \left(\frac{(a/a_{\text{eq}})^2}{x^{(0)}(1-y_p/2)}\right)^{-1} (1-f_\nu)^2 (\Omega_0 h^2)^2 (\Omega_b h^2) \text{ Mpc}^{-1}, \quad (\text{B22})$$

$$M_{\text{BR}} = 6.75 \times 10^{-1} \left(\frac{(a/a_{\text{eq}})^2}{x^{(0)}(1-y_p/2)}\right)^3 (1-f_\nu)^{-6} (\Omega_0 h^2)^{-6} (\Omega_b h^2)^{-2} M_\odot. \quad (\text{B23})$$

6. γ transition epoch after the recombination

The adiabatic index γ after the recombination is changed when the Compton energy transfer time is equal to the sound oscillation time scale. We define this scale by $1/k_c^{\text{comv}} = c_e \eta_E$, which leads

$$k_c^{\text{comv}} = \frac{6.08 \times 10^{14} x^{(0)} (1-y_p/2) (1-f_\nu)^3}{(\mu^3 T_e)^{1/2} (a/a_{\text{eq}})^3} (\Omega_0 h^2)^3 \text{ Mpc}^{-1}, \quad (\text{B24})$$

$$M_c = \frac{1.60 \times 10^{-31} (\mu^3 T_e)^{3/2} (a/a_{\text{eq}})^9}{x^{(0)3} (1-y_p/2)^3 (1-f_\nu)^9} \Omega_b h^2 (\Omega_0 h^2)^{-9} M_\odot, \quad (\text{B25})$$

where T_b is the matter temperature in unit of Kelvin as is mentioned above.

APPENDIX C: MATTER TEMPERATURE

As we have derived in §V, the matter temperature follows

$$T_b^{(0)'} + 2\frac{a'}{a}T_b^{(0)} = \eta_E^{-1}(T_\gamma^{(0)} - T_b^{(0)}), \quad (C1)$$

where $T_\gamma^{(0)}$ is the photon temperature, and the Compton energy transfer time scale η_E is defined as

$$\eta_E^{-1} = \frac{8}{3} \frac{a}{a_0} \frac{x^{(0)}(1 - y_p/2)\sigma_T\rho_\gamma^{(0)}}{m_e(1 + x^{(0)} - (x^{(0)} + 3/2)y_p/2)}. \quad (C2)$$

The formal solution is

$$T_b^{(0)} = \frac{1}{a^2} \int_0^\eta d\eta' a^2 T_\gamma^{(0)} \eta_E^{-1} \exp\left[-\int_{\eta'}^\eta d\eta'' \eta_E^{-1}\right]. \quad (C3)$$

We can obtain the epoch when the matter temperature deviates from the photon temperature as follows. This epoch is naturally defined as the epoch when the function $\eta_E^{-1} \exp(-\int_{\eta'}^\eta d\eta'' \eta_E^{-1})$ takes its maximum value. Thus we need to solve the equation $\dot{\eta}_E^{-1} + (\eta_E^{-1})^2 = 0$. From Eq. (C2),

$$\eta_E^{-1} = 1.8 \times 10^8 (1 - f_\nu)^3 (\Omega_0 h^2)^3 \frac{x^{(0)}(1 - y_p/2)}{(1 - 3y_p/4)} \left(\frac{a}{a_{\text{eq}}}\right)^{-3} \text{Mpc}^{-1}. \quad (C4)$$

Therefore we get the epoch as

$$\frac{a}{a_{\text{eq}}} = 2.3 \times 10^3 x^{(0)2/5} (\Omega_0 h^2)^{4/5}, \quad (C5)$$

with neglecting the time variation of the ionization fraction $x^{(0)}$. Here we set $y_p = 0.23$ and $f_\nu = 0.405$.

We approximate the fraction of the residual electron as [21]

$$x^{(0)} \simeq 10^{-5} (\Omega_0 h^2)^{1/2} (\Omega_b h^2)^{-1}. \quad (C6)$$

Then from Eq.(C5), we conclude that the matter temperature decouples from the photon's at $(1 + z) \simeq 1000(\Omega_b h^2)^{2/5}$. After this epoch, the matter temperature is adiabatically cooling.

According to the fully numerical calculation [12], the matter temperature in this adiabatic cooling phase is well reproduced by the formula,

$$T_b^{(0)} = 4.5 \times 10^{-3} (1 + z)^2 (\Omega_b h^2)^{-2/5} \text{K}. \quad (C7)$$